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A CHVATAL-ERDOS TYPE CONDITION FOR PANCYCLABILITY

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A Chvátal-Erdős type condition for pancyclability

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Abstract

Let G be a graph and S a subset of $V(G)$. Let $\alpha(S)$ denote the maximum number of pairwise nonadjacent vertices in the subgraph $G < S >$ of G induced by S. If $G < S >$ is not complete, let $\kappa(S)$ denote the smallest number of vertices separating two vertices of S and $\kappa(S) = |S| - 1$ otherwise. We prove that if $\alpha(S) \leq \kappa(S)$ and $|S|$ is large enough (depending on $\alpha(S)$), then G is S-pancyclable, that is contains cycles with exactly p vertices of S for every p, $3 \leq p \leq |S|$. This is a generalization of the result of Flandrin, Li, Marczyk, Woźniak and Schiermeyer stating that a graph G of order n that satisfies the Chvátal-Erdős condition $\alpha(G) \leq \kappa(G)$ is pancyclic provided n is sufficiently large with respect to $\alpha(G)$.

Résumé

Soit G un graphe et S un sous-ensemble de $V(G)$. Soit $\alpha(S)$ le nombre maximum de sommets deux à deux non adjacents dans $G < S >$, sous-graphe de G induit par S. Si $G < S >$ n'est pas complet, soit $\kappa(S)$ le plus petit nombre de sommets séparant deux sommets de S et $\kappa(S) = |S| - 1$ sinon. On démontre que si $\alpha(S) \leq \kappa(S)$ et $|S|$ est assez grand (par rapport à $\alpha(S)$), alors G est S-pancyclable, c'est à dire possède des cycles contenant exactement p sommets de S, $3 \leq p \leq |S|$. C'est une généralisation d'un résultat de Flandrin, Li, Marczyk, Woźniak et Schiermeyer qui établit qu'un graphe G d'ordre n satifaisant la condition de Chvátal-Erdős $\alpha(G) \leq \kappa(G)$ est pancyclique pourvu qu'il soit d'ordre suffisamment grand par rapport à $\alpha(G)$.

Keywords: hamiltonian graphs, pancyclic graphs, cycles, connectivity, stability number, cyclability, pancyclability.
AMS Classification: 05C38, 05C45.

1 Introduction

For a graph G we denote by $V = V(G)$ its vertex-set and by $E = E(G)$ its set of edges. The symbols $\alpha = \alpha(G)$ and $\kappa = \kappa(G)$ stand for the *stability number* and the *connectivity* of G.

If S is a subset of V, $G < S >$ is the subgraph of G induced by S and $\alpha(S)$ denotes the maximum number of pairwise nonadjacent vertices in S. If $G < S >$ is not complete, we define $\kappa(S)$ as the smallest number of vertices separating two vertices of S and we put $\kappa(S) = |S| - 1$ otherwise.

A vertex of S is called an *S-vertex* and a cycle of G that contains exactly p S-vertices is said to have *S-length p*; such a cycle will be denoted by C_p^S. The vertex set S is said to be *cyclable in* G if G contains a cycle through all the vertices of S and *pancyclable in* G if contains cycles of every S-length p with $3 \leq p \leq |S|$.

Notice that putting $S = V(G)$ in the above definitions concerning S we clearly get back the usual notions of stability number, connectivity, hamiltonicity and pancyclicity.

Let us recall the notion of Ramsey numbers $R(k, m)$ that we need to express our main result.
Given two integers $k \geq 2$ and $m \geq 2$, the *Ramsey number,* $R(k, m)$, is the smallest integer such that each graph of order $n \geq R(k, m)$ contains a clique on k vertices or a stable set of cardinality m. The existence of such a number is guarantee by the famous Ramsey's theorem (see [9]).

In 1971 Bondy [2] suggested that almost all nontrivial sufficient conditions for a graph to be hamiltonian also imply that it is pancyclic except for maybe a simple family of graphs.

This "metaconjecture" of Bondy was at the origin of many results on hamiltonicity and pancyclicity. Here we will need the well known Chvátal-Erdős theorem.

Theorem 1 (Chvátal, Erdős [5]) *Let G be a graph of order at least 3 satisfying $\alpha \leq \kappa$. Then G is hamiltonian.*

Note that for Chvátal-Erdős condition $\alpha \leq \kappa$ the metaconjecture does not hold because there is a large family of triangle-free graphs (see for example the survey [4]) that satisfy the Chvátal-Erdős condition but are clearly not pancyclic. However, if we add the assumption that the order of G is large enough with respect to the stability number of the graph, the Chvátal-Erdős condition happens to be sufficient for pancyclicity. More precisely, in [6], the authors proved the following.

Theorem 2 (Flandrin, H. Li, Marczyk, Schiermeyer, Woźniak [6]) *Let G be k-connected graph with stability number α. If $\alpha \leq k$ and the order of G is at least $2R(4\alpha, \alpha+1)$, then G is pancyclic.*

We now raise the question of the existence of some analogous nontrivial results when we consider only a subset S of V and the parameters $\alpha(S)$ and $\kappa(S)$ instead of α and κ, and S-cyclability and S-pancyclability instead of hamiltonicity and pancyclicity.

Let us recall those concerning cyclability, first by I. Fournier ([8]) and then improved in [1] and [7].

Theorem 3 (Fournier [8]) *Let G be a 2-connected graph and $S \subset V$. If $\alpha(S) \leq \kappa$, then S is cyclable in G.*

Theorem 4 (Broersma, H. Li, J. Li , Tian, Veldman [1]) *Let G be a graph and S a subset of $V(G)$ with $|S| \geq 3$. If $\alpha(S) \leq \kappa(S)$, then S is cyclable in G.*

Actually, in [1] it is shown that if G is 2-connected and $\alpha(S) \leq \kappa(S)$, then the same conclusion holds. However, with the simple modification of the proof (see [7]) we can easily get the last result.

In this paper we give an extension of Theorem 2 and prove that the above condition also implies that S is pancyclable in G provided the cardinality of S is large enough with respect to $\alpha(S)$.

Theorem 5 *Let G be a graph and $S \subset V$. If $\alpha(S) \leq \kappa(S)$ and $|S| \geq 2R(4\alpha(S), \alpha(S)+1)$, then S is pancyclable in G.*

2 Notations

We use Bondy and Murty's book [3] for terminology and notation not defined here and consider only finite, undirected and simple graphs.

For a graph G, a vertex x in V and a subgraph H in G, $N_H(x)$ denotes the set of the neighbors of x in H and the *degree*, $d_H(x)$, of x with respect to H is equal to $|N_H(x)|$. When $H = G$, the subscript H will be omitted.

Let C be a cycle in G with an arbitrary orientation and x and y two vertices of C. The *segment* $C[x,y]$ is the subpath of C from x to y according to the orientation (x and y included). We define in a similar way the segment $P[x,y]$ of a path P with a given orientation.

We also use the notations x^+ and x^- for the successor and the predecessor of x on C. If considering a subset S of $V(G)$ and two S-vertices s_1 and s_2 on C, s_2 is said to be the S-vertex following s_1 on C if $C[s_1,s_2] \cap S = \{s_1, s_2\}$. We say that s_1 and s_2 are S-consecutive on C.

3 Proof of Theorem 5

Suppose that G is a graph, S a subset of $V(G)$ such that $\alpha(S)$ and $\kappa(S)$ satisfy $\alpha(S) \leq \kappa(S)$ and $|S| \geq 2R(4\alpha(S), \alpha(S)+1) \geq 8$. Notice that if $\alpha(S) = 1$, then S is a clique and we are done, therefore we can assume $2 \leq \alpha(S) \leq \kappa(S)$.

The proof will be divided into two parts, depending on the S-length of the cycles we want to obtain.

CASE 1 : G contains a C_p^S for each $p \geq \frac{|S|}{2} - 1$.

Observe that, by Theorem 4 and a result of Flandrin et al. [7], this statement is evident for $p = |S|$ and suppose that G contains a cycle C_p^S with $p \geq \frac{|S|}{2}$. We shall prove that G also contains a C_{p-1}^S.

Let $a_1, a_2, \ldots, a_p$ be the vertices of $C_p^S \cap S$ appearing in that order on C_p^S, where the indices are considered modulo p. Since $p \geq \frac{|S|}{2} \geq R(4\alpha(S), \alpha(S)+1)$, and the graph induced by $C_p^S \cap S$ has no stable set of cardinality $\alpha(S)+1$, it follows from the Ramsey's theorem that it contains a clique, say K, having $4\alpha(S)$ S-vertices. Assume that among the cycles of S-length p passing through $\{a_1, a_2, \ldots, a_p\}$, C_p^S is chosen such that it contains as many edges of K as possible and fix an arbitrary orientation of C_p^S.

Suppose now that G does not contain any cycle with $p-1$ S-vertices. Clearly a_i cannot be adjacent to a_{i+2} for $1 \leq i \leq p$ and, consequently, if a_i belongs to K, a_{i+2} is not in K.

Let $d_1, d_2, ..., d_r$ be the vertices of K, appearing in that order on C_p^S, such that for $1 \leq i \leq r$, the S-vertex following d_i on C_p^S is not in K.

¿From the above remark, there are at least $2\alpha(S)$ such vertices d_i, and we shall denote by b_i the S-vertex following d_i on C_p^S, $1 \leq i \leq r$, $r \geq 2\alpha(S)$. Since $2\alpha(S) > \alpha(S)$, there are necessarily two vertices b_{i_1} and b_{i_2} that are adjacent.

Using the edges $b_{i_1}b_{i_2}$ and $d_{i_1}d_{i_2}$, we easily obtain a cycle with exactly the same S-vertices than C_p^S and that contains more edges of K than C_p^S, and we get a contradiction with the choice of C_p^S. This implies the existence of a cycle of S-length $p-1$ as soon as $p \geq \frac{|S|}{2}$. Hence, by induction, G contains cycles C_p^S for each $p \geq \frac{|S|}{2} - 1$.

CASE 2 : G contains a C_p^S for each $p < \frac{|S|}{2} - 1$.

Since $|S| \geq 2R(4\alpha(S), \alpha(S)+1)$ and S has no stable set of cardinality $\alpha(S)+1$, it follows from Ramsey theorem that S contains a clique on $4\alpha(S)$ vertices. Thus, our statement is is evident for $3 \leq p \leq 4\alpha(S)$. Suppose G has a C_p^S for some p satisfying $p < \frac{|S|}{2} + 1 - 4\alpha(S)$. We claim that it contains also a cycle with exactly $p + 4\alpha(S) - 2$ S-vertices.

Since $p = |C_p^S \cap S| < \frac{|S|}{2}$, the graph $G - C_p^S$ contains at least $\frac{|S|}{2} \geq R(4\alpha(S), \alpha(S)+1)$ S-vertices, whence also contains a clique, say K, on $4\alpha(S)$ vertices.

By Menger's theorem there are at least $\min(\kappa(S), p, 4\alpha(S))$ vertex-disjoint paths between the vertices of K and the vertices of $C_p^S \cap S$. Consequently, using the assumptions $\alpha(S) \leq \kappa(S)$, there exist r $=\min(\alpha(S), p)$ vertex-disjoint paths, that join C_p^S with K. Fix an arbitrary orientation of C_p^S, and denote by x_i and y_i, $i = 1, 2, \ldots, r$ the end-vertices of those paths belonging to $V(C_p^S)$ and $V(K)$, resp. We assume that the vertices $x_1, x_2, ..., x_r$ appear on the cycle C_p^S in the order of their indices. Let P_i $(i = 1, 2, \ldots, r)$ be the path of end vertices x_i and y_i. Notice that x_i does not belong necessarily to S. We will assume that every path P_i has minimum S-length, whence, from the definition of $\alpha(S)$, $|V(P_i) \cap S| \leq 2\alpha(S)$ for every P_i, $1 \leq i \leq r$. Set $l_i = |(V(P_i) - \{x_i\}) \cap S| \leq 2\alpha(S)$, $i = 1, 2, \ldots, r$.

Claim 1 *Assume that if for some i, $1 \leq i \leq r$, we have $C_p^S[x_i, x_{i+1}] \cap S \subset \{x_i, x_{i+1}\}$. Then G contains a $C_{p+4\alpha(S)-2}^S$.*

Proof. Suppose first that $l_i + l_{i+1} \leq 4\alpha(S) - 2$. Delete the interior vertices and the edges of the segment $C_p^S[x_i, x_{i+1}]$ and add the paths P_i, P_{i+1} and Q_i, where Q_i is a path from y_i to y_{i+1} in K with $4\alpha(S)-2-l_i-l_{i+1} \geq 0$ interior vertices. In this way we obtain cycle with $p+4\alpha(S)-2$ vertices of S.

Suppose now that $4\alpha(S)-1 \leq l_i + l_{i+1} \leq 4\alpha(S)$ and consider the case $l_i = 2\alpha(S)$ and $l_{i+1} = 2\alpha(S)-1$. Let $s_1, s_2, \ldots, s_{2\alpha(S)} = y_i$ be the S-vertices of the directed path $P_i[x_i, y_i]$ appearing on $P_i[x_i, y_i]$ in the order of their indices. Obviously, $s_1 \notin V(C_p^S)$. Because of the choice of P_i, the set $s_2, s_4, s_6, \ldots, s_{2\alpha(S)}$ is stable. Denote now by z the last S-vertex on C_p^S (according to the orientation of C_p^S) before x_i. From the definition of $\alpha(S)$, z must be adjacent to a vertex s_{2j} for some $j \leq \alpha(S)$. Delete the interior vertices and the edges of the segment $C_p^S[z, x_{i+1}]$ and add the edge zs_{2j} and the paths $P_i[s_{2j}, y_i]$, P_{i+1} and Q_i, where Q_i is a path from y_i to y_{i+1} in K with $4\alpha(S)-2-(2\alpha(S)-2j+1)-(2\alpha(S)-1) \geq 0$ interior vertices. In this way we get a cycle having $p+4\alpha(S)-2$ S-vertices as required. Considering, if necessary, the first S-vertex on C_p^S after x_{i+1}, we proceed in the similar way in other subcases of the case $4\alpha(S)-1 \leq l_i + l_{i+1} \leq 4\alpha(S)$. ■

Consequently, we assume that any two vertices x_i and x_{i+1} are separated by at least one S-vertex on C_p^S. There are two possibilities, depending on the relative value of p with respect to $\alpha(S)$.

Case 2.1 : $\alpha(S) \leq p$

We have $r = \alpha(S)$. For $1 \leq i \leq \alpha(S)$, let v_i be the S-vertex following x_i on C_p^S, which is, from our hypothesis, interior to the segment $C_p^S[x_i, x_{i+1}]$. Let x be any vertex of $K \setminus \{y_1, y_2, \ldots, y_r\}$. Then $A = \{v_1, v_2, ..., v_{\alpha(S)}, x\}$ is a subset of S with $\alpha(S)+1$ vertices and so the subgraph $G < A >$ contains at least one edge. Suppose first that $xv_i \in E$ for some i. Then we apply Claim 1, where the path P_{i+1} is replaced by he path x, v_i and we obtain a cycle having $p+4\alpha(S)-2$ vertices of S.

So we may assume now that such an edge joins two vertices of the cycle C_p^S, say v_i and v_j (see Fig. 1). Suppose that $l_i + l_j \leq 4\alpha(S)-2$. Delete the interior vertices and the edges of the segments $C_p^S[x_i, v_i]$, $C_p^S[x_j, v_j]$ and add the paths P_i, P_j and Q_{ij}, where Q_{ij} is a path from y_i to y_j in K with $4\alpha(S)-2-l_i-l_j \geq 0$ interior vertices. In this way we obtain cycle with $p+4\alpha(S)-2$ vertices of S. It remains the case when $4\alpha(S)-1 \leq l_i + l_j \leq 4\alpha(S)$. Suppose $l_i = 2\alpha(S)$ and $l_j = 2\alpha(S)-1$ and let $s_1, s_2, \ldots, s_{2\alpha(S)} = y_i$

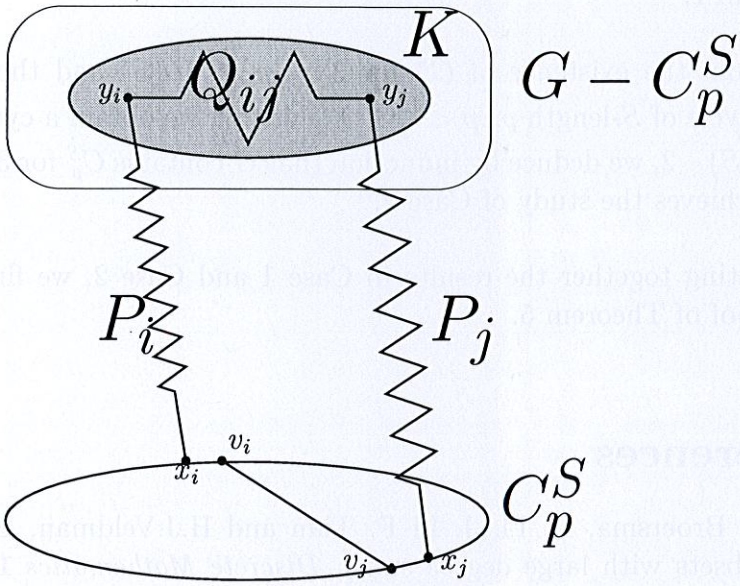

Figure 1:

be the S-vertices of the directed path $P_i[x_i, y_i]$ appearing on $P_i[x_i, y_i]$ in the order of their indices. Clearly, $s_1 \notin V(C_p^S)$. Denote now by z the last S-vertex on C_p^S (according to the orientation of C_p^S) before x_i. We can show as in the proof of Claim 1 that z must be adjacent to a vertex s_{2m} for some $m \leq \alpha(S)$. Delete the interior vertices and the edges of the segments $C_p^S[z, v_i]$, $C_p^S[x_j, v_j]$ and add the edge zs_{2m} and the paths $P_i[s_{2m}, y_i]$, P_j and Q_{ij}, where Q_{ij} is a path from y_i to y_j in K with $4\alpha(S) - 2 - (2\alpha(S) - 2m + 1) - (2\alpha(S) - 1) \geq 0$ interior vertices. Thus, we get a cycle having $p + 4\alpha(S) - 2$ vertices of S as required. We proceed in the similar way in other subcases of the case $4\alpha(S) - 1 \leq l_i + l_j \leq 4\alpha(S)$.

Case 2.2 : $p < \alpha(S)$

We have $r = p$. If one of the segments $C[x_i, x_{i+1}]$ has no interior vertex in S then, by Claim 1, we are done. Otherwise, there is exactly one vertex of S interior to the segment $C[x_i, x_{i+1}]$ for $1 \leq i \leq p$. If $l_i + l_{i+1} \leq 4\alpha(S) - 1$, for some i, then the cycle $x_i^-, x_i, P_i[x_i, y_i], Q_i[y_i, y_{i+1}], P_{i+1}[y_{i+1}, x_{i+1}], x_{i+1}, x_{i+1}^+, \ldots, x_i^-$ i has S-length $p + 4\alpha(S) - 2$, where Q_i is a path from y_i to y_{i+1} in K with $4\alpha(S) - 2 - l_i - l_{i+1} + 1 \geq 0$ interior vertices. If $l_i + l_{i+1} = 4\alpha(S)$ we proceed as in the proof of Claim 1.

¿From the existence of C_p^S for $3 \leq p \leq 4\alpha(S)$ and the fact that for every cycle of S-length p, $p < \frac{|S|}{2} + 1 - 4\alpha(S)$, we obtain a cycle of S-length $p + 4\alpha(S) - 2$, we deduce by induction that G contains C_p^S for $3 \leq p < \frac{|S|}{2} - 1$. This achieves the study of Case 2.

Putting together the results in Case 1 and Case 2, we finally complete the proof of Theorem 5.

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