Endpoint Extendible Paths in Dense Graphs

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Abstract

Let G be a graph of order n. A path P of G is *extendible* if it can be extended to a longer path from one of its two endvertices, otherwise we say P is *non-extendible*. Let G be a graph of order n. We show that there exists a threshold number s such that every path of order smaller than s is extendible and there exists a non-extendible path of order t for each $t \in \{s, s+1, \cdots, n\}$ provided G satisfies one of the following three conditions:

- $d(u) + d(v) \geq n$ for any two of nonadjacent vertices u and v.
- G is a P_4-free 1-tough graph.
- G is a connected, locally connected, and $K_{1,3}$-free graph.

1 Introduction

We generally follow the notation of Chartrand and Lesniak [1]. All graphs considered in this paper are simple finite graphs, i.e., graphs with finite number of vertices, without loops, and without multiple edges. Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. We write $G = (V(G), E(G))$. For convenience, we write V instead of $V(G)$ and E instead of $E(G)$ if the referred graph G

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is well understood. We call $|G| = |V(G)|$ the order of G. In this paper, we reserve n for the order of G. We define that

$$N_H(v) = \{w \in V(H) \ : \ vw \in E(G)\} \quad \text{and} \quad d_H(v) = |N_H(v)|,$$

where $v \in V$ and H is a subgraph of G. For convenience, let $N(v) = N_G(v)$, $d(v) = d_G(v)$, $\delta = \min\{d(v) \ : \ v \in V\}$ the minimum degree of G, and

$$\sigma_2(G) = \min\{d(u) + d(v) \ : \ \ uv \notin E\}.$$

Moreover, we define $N[v] := N(v) \cup \{v\}$ and name it the close neighborhood of v for each $v \in V(G)$. A subgraph H of G is called an *induced subgraph* of G if there exists an $X \subseteq V(G)$ such that $H = G[X]$. Let $G - X = G[V - X]$. For any two vertices u, $v \in V$ and an induced subgraph H of G, let uHv denote an arbitrary shortest path connecting u to v in $G[V(H) \cup \{u, v\}]$ if it is connected.

For any positive integer k, let P_k denote a path of order k. A graph G is *traceable* if it contains a hamiltonian path. Ore [13] proved that a graph G of order n is traceable if $\sigma_2(G) \geq n-1$. A graph G is *path extendible* if for every path P_k with $k < |G|$ there exists a path P_{k+1} such that $V(P_k) \subseteq V(P_{k+1})$. Clearly, if a graph is path extendible then it is traceable. Hendry [6] showed that most known sufficient conditions for traceable graphs are also sufficient for extendible. In particular, he showed that a graph G of order n is path extendible if $\sigma_2(G) \geq n - 1$.

A path P of a graph G is called a *maximum path* if $|P| \geq |Q|$ for all paths Q in G. However, there are two different versions of *maximal paths*: vertex version and edge version. A path P is *vertex-maximal* if there does not exist a path Q such that $V(P)$ is a proper subset of $V(Q)$. A path P is called *edge-maximal* if there does not exist a path Q such that $E(P)$ is a proper subset of $E(Q)$. Clearly, all vertex-maximal paths are edge-maximal paths. On the other hand, not all edge-maximal paths are vertex-maximal paths. If G is a traceable graph, all vertex-maximal paths are hamiltonian paths. However, many nontrivial problems arise naturally for edge-maximal paths. In this paper, we will only consider edge-maximal paths. To avoid cumbersome notation, we simply call them *maximal paths*. We notice that the following statements are equivalent:

- P is a maximal path.
- P cannot be extended from one of its two endvertices.
- No path contains P as a proper subpath.

For any graph G, we let $S(G) = \{|P| \ : \ P \text{ is a maximal path of } G\}$ and call it the *maximal-path spectrum* of G. We also say that G *realizes* $S(G)$. A set S of positive integers is called a *path spectrum* if there exists a **connected** graph G such that $S(G) = S$. Note that the condition of connectivity is important. Otherwise, all positive integer sets would be path spectra.

The ultimate goal is to establish an efficient algorithm to determine whether a set S of positive integers is a path spectrum. So far, there have been very little progress on this direction. Connected graphs G with $|S(G)| = 1$ are investigated by Thomassen [14] and, independently, by Jacobson et al [9, 10]. Jacobson et al [8] show that all pairs of positive integers are path spectra. Chen et al [2] show that there are infinitely many k-sets of positive integers which are not path spectra for each integer $k \geq 3$. Chen et al [3] studied path spectra for trees.

In this paper, we investigate graphs G such that $S(G)$ consists of consecutive positive integers. A graph G is called a *string path spectrum* graph (*SPS-graph*) if there exists a positive integer s such that $S(G) = \{s, s+1, \ldots, n\}$, where n is the order of G. Clearly, all SPS-graphs are traceable. The property of SPS-graphs are much stronger than traceable graphs. We will show some well-known sufficient conditions for traceable graphs are sufficient for SPS-graphs. The following one is an Ore-type result.

Theorem 1 *Let G be a graph of order n. If $\sigma_2(G) \geq n$, then G is an SPS-graph.*

A vertex set X of a graph G is a *cut* if $G - X$ has more components than G does. A cut X of G is minimal if there does not exist a cut Y of G such that Y is a proper subset of X. A minimal cut X of G is called a *skew-cut* if $G - X$ is a union of two disjoint cliques A and B such that $N(x) \supset V(B)$ for each $x \in X$ and $N_A(x)$'s, for all $x \in X$, form a partition of A. Notice that, in the above definition, $d(a) + d(b) = n - 1$ for each $a \in A$ and $b \in B$. Clearly, if X is a skew-cut of G then it is the unique skew-cut of G. A graph G is named a *skew-joint* graph if it has a skew-cut. An example of a skew-joint graph is shown in Figure 1.

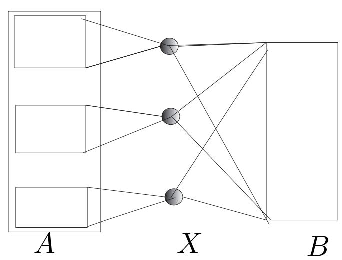

Figure 1: A Skew Joint Graph with $|X| = 3$

Theorem 2 *Let G be a graph of order n. If $\sigma_2(G) \geq n-1$, then G is either an SPS-graph or a skew-joint graph.*

Some skew-joint graphs are SPS-graphs while others are not. We believe it is very difficult to character these skew-joint graphs which are SPS-graphs. In the following, we will discuss path spectra of some special skew-joint graphs. Let G be a skew-joint graph of order n, X be the skew-cut of G, and A and B be the two disjoint cliques of $G - X$ such that $N(x) \supset V(B)$ for each $x \in X$ and $N_A(x)$'s, for all $x \in X$, form a partition of A. Let $a = |A|$, $b = |B|$, and $s = |X|$. Clearly $a \geq s$. We will only consider the case that X is an independent set, $b \geq s+1 \geq 4$, and $|N_A(x)| \geq 3$ for every $x \in X$. Let x_1, x_2, $\cdots$, x_s be a list of all vertices of X such that $d_A(x_1) \leq d_A(x_2) \leq \cdots \leq d_A(x_s)$. For convenience, let $a_1 = d_A(x_1)$ and $a_2 = d_A(x_2)$, i.e. a_1 is the smallest value of $d_A(x)$ and a_2 is the second smallest value of $d_A(x)$ for all $x \in X$. Notice that $a_2 = a_1$ may happen. We assume that $a_1 \geq 3$. Let $P[u, v]$ be a maximal path of G. We observe the following facts:

- If $u \in A$ and $v \in A$, then $|V(P)| \geq |N[u] \cup N[v]| \geq a+1$.
- If $u \in A$ and $v \in X$, then $V(P) \supseteq N[u] \cup N[v] \supseteq A \cup B \cup \{v\}$. Since $P - v$ is connected, $V(P) \cap X \neq \{v\}$. So, $|V(P)| \geq a+b+2$.
- If $u \in A$ and $v \in B$. then $V(P) \supseteq N[u] \cup N[v] = A \cup X \cup B = V(G)$, so that$|V(P)| = n$.
- If $u \in X$ and $v \in X$, then $V(P) \supseteq N_A(u) \cup N_A(v) \cup B \cup \{u, v\}$. Since $P - u - v$ is connected, there exists a vertex $w \in X \cap (V(P) - \{u, v\})$ so that $N_A(w) \cap V(P) \neq \emptyset$. So, $|V(P)| \geq a_1 + a_2 + b + 4$. On the other hand, since $G[A]$ is a clique and $N(x) \supseteq B$ for each $x \in X$, there exists a maximal path $P[u, v]$ such that $u \in X$ and $v \in X$ and $|V(P)| = \ell$ for each $\ell \geq a_1 + a_2 + b + 4$ as shown in Figure 2.

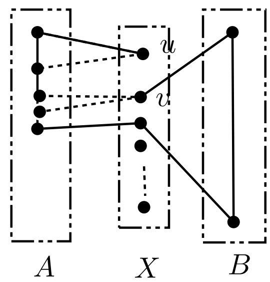

Figure 2: A maximal path connecting $u \in X$ and $v \in X$

- If $u \in X$ and $v \in B$, then $V(P) \supseteq N_A(u) \cup X \cup B$. Since $P - u$ is connected, there exists a vertex $w \in V(P) \cap A$ that is adjacent to some vertex of $X - \{u\}$. Clearly, $w \notin N_A(u)$. So, $|V(P)| \geq a_1 + s + b + 1$.
- If $u \in B$ and $v \in B$, then $V(P) \supseteq N[u] \cup N[v] \supseteq X \cup B$, so that $|V(P)| \geq s + b$. Moreover, since $N(x)$'s form a partition of A, $|V(P) \cap A| \geq 2$ if $V(P) \cap A \neq \emptyset$. So, $|V(P)| \neq b + s + 1$. On the other hand, there is a maximal path $P[u, v]$ such that $|V(P)| = \ell$ for $\ell = b + s$ or $\ell \geq s + b + 2$ as shown in Figure 3.

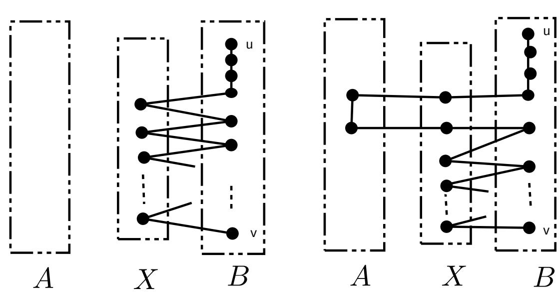

Figure 3: Maximal paths connecting $u \in B$ and $v \in B$

If $a_1 + a_2 \leq \min\{s-4, a-b-3\}$, then $S(G) = \{a_1 + a_2 + b + 4, a_1 + a_2 + b + 5, \cdots, n\}$, which implies that G is an SPS-graph.

If $a_1 + a_2 \geq s + b$, $S(G) = \{s+b, s+b+2, s+b+3, \cdots, n\}$, which shows that G is not an SPS-graph.

Let F be a graph. A graph G is F-*free* if G does not contain F as an *induced subgraph.* The toughness of a non-complete graph G is defined by Chvátal [4] as

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} \quad : \quad S \text{ is a cut of } G\},$$

where $\omega(G-S)$ is the number of components of $G-S$. For a complete graph K_n, define $t(K_n) = \infty$. A graph G is t-*tough* if $t(G) \geq t$. Clearly, if G is hamiltonian, then G is 1-tough. However, the converse is not true. Chvátal [4] conjectured there is a universal constant t_0 such that all t_0-tough graphs are hamiltonian. Although the conjecture is still open, many results have been obtained along this direction. Jung [11] proved a complicated result on hamiltonian graphs, which implies the following result.

Theorem 3 (Jung) *If G is a P_4-free and 1-tough graph of order $n \geq 3$, then G is hamiltonian.*

We will prove a similar result as follows.

Theorem 4 *If G is a P_4-free and 1-tough graph, then G is an SPS-graph.*

Another classic result for hamiltonian graphs involving forbidden subgraphs is due to Oberly and Sumner [12].

Theorem 5 (Oberly and Sumner) *If G is a connected, locally connected, and $K_{1,3}$-free graph, then G is hamiltonian.*

We obtain a similar result as follows.

Theorem 6 *If G is a connected, locally connected, and $K_{1,3}$-free graph, then G is an SPS-graph.*

Due to lengths of those proofs, the proofs of Theorems 2, 4 and 6 will be placed to sections 2, 3 and 4, respectively. Let $P = P[u,v]$ be a path of G. We always assume $P[u,v]$ has an orientation from u to v. For any $x \in V(P[u,v])$, let x^+ denote the successor of x along P if $x \neq v$, and x^- be the predecessor of x along P if $x \neq u$. Let $x^{++} = (x^+)^+$ if $x \neq v$, or v^- and $x^{--} = (x^-)^-$ if $x \neq u$, or u^+. For an $S \subseteq V(P[u,v])$, we define

$$\begin{aligned} S^+ &= \{x^+ : x \in S\} \quad \text{if } v \notin S, \\ S^- &= \{x^- : x \in S\} \quad \text{if } u \notin S. \end{aligned}$$

2 Proof of Theorem 2

Let G be a graph of order n such that $\sigma_2(G) \geq n-1$. Suppose, to the contrary, G is neither an SPS-graph nor a skew-joint graph. Since $\sigma_2(G) \geq n-1$, G is connected. If the connectivity $\kappa(G) = 1$, then G is a union of two cliques with exactly one vertex in common, so it is a skew-joint graph, a contradiction. Therefore, G is 2-connected.

Since G is not an SPS-graph, there exists a positive integer p such that G contains a maximal path $P = P[u, v]$ of order p and G does not contain a maximal path of order $p+1$. Since G is traceable, $p \leq n-2$. Let $H = G - V(P)$ and $h = |V(H)|$. Since $h+p = n$, we have the following claim.

Claim 1 $h \geq 2$.

Let

$$\begin{aligned} Y_1 &= \{y \in V(H) \ : \ d_P(y) \geq 1\}, \\ Y_2 &= \{y \in V(H) \ : \ d_P(y) \geq 2\}, \\ X_1 &= N_P(Y_1) = \{x_1, x_2, \cdots, x_s\}, \quad \text{and} \\ X_2 &= N_P(Y_2). \end{aligned}$$

Clearly, $Y_2 \subseteq Y_1$, $X_1 = N_P(H)$, and $X_2 \subseteq X_1$. Without loss of generality, we assume that x_1, x_2, $\cdots$, x_s are listed in the order along the orientation of $P[u, v]$ from u to v. We will show $Y_2 = \emptyset$ by sequence of claims.

Since $P[u, v]$ is a maximal path, we have $N(u) \subseteq V(P[u, v])$ and $N(v) \subseteq V(P[u, v])$, so that $u, v \notin X_1$. By the definition of x_1, we have that $N_H(x_1^-) = \emptyset$. Let $y_1 \in Y_1 \subseteq V(H)$ such that $x_1y_1 \in E$. We claim $N(x_1^-) \cap N_P^-(y_1) = \emptyset$. Otherwise, suppose there exist a $w \in N_P(y_1)$ such that $x_1^- w^- \in E$. Then, $P[u, x_1^-]P^-[w^-, x_1]y_1P[w, v]$ is a maximal path of order $p+1$, a contradiction. Therefore,

$$d_P(x_1^-) + d_P(y_1) = |N_P(x_1^-)| + |N_P^-(y_1)| = |N_P(x_1^-) \cup N_P^-(y_1)| \leq |V(P)|.$$

Since $d_H(x_1^-) = 0$,

$$n - 1 \leq d(x_1^-) + d(y_1) = d_P(x_1^-) + d_P(y_1) + d_H(x_1^-) + d_H(y_1) \leq p + h - 1 = n - 1.$$

Then, the above equalities hold, so that $N_P(x_1^-) = V(P) - N_P^-(y_1)$ and $N_H(y_1) = V(H) - \{y_1\}$. In particular, H is connected.

We claim that $N(u) \cap N_P^-(y_1) \subseteq \{x_1^-\}$. Otherwise, let $w \in N_P^-(y_1) - \{x_1^-\}$ such that $uw \in E$. Path $P^-[x_1^-, u]P^-[w, x_1]y_1P[w^+, v]$ is a maximal path of order $p+1$, a contradiction. Then, similar to the previous paragraph, we can show that $N[u] - \{x_1^-\} = V(P) - N_P^-(y_1)$.

Claim 2 $h \geq 3$.

Proof: Suppose, to the contrary, $h = 2$. Let $V(H) = \{y_1, y_2\}$, where $x_1y_1 \in E$. Since $N(x_1^-) = V(P) - N_P^-(y_1)$ and $v \notin N_P(y_1)$, we have $x_1^- v^- \in E(G)$. If $uv \notin E(G)$, $P[u, x_1^-]P^-[v^-, x_1]y_1y_2$ is a maximal path of order $p+1$, a contradiction. Thus, $uv \in E(G)$.

If $d_P(y_1) \geq 2$, let $x_k \neq x_1 \in N_P(y_1)$. If $vx_k^- \in E(G)$, G contains a maximal path $P[u, x_1^-]P^-[v^-, x_k]y_1P[x_1, x_k^-]v$ of order $p+1$, a contradiction. Thus, $vx_k^- \notin E(G)$. Since $N[u] - \{x_1^-\} = V(P) - N_P^-(y_1)$, $uv^- \in E(G)$. So $P^-[x_k^-, u]P^-[v^-, x_k]y_1y_2$ is a maximal path of order $p+1$, a contradiction. Thus, $d_P(y_1) = 1$, so that $y_2x_s \in E$. Similarly, we can show that $d_P(y_2) = 1$. Since $h = 2$, we have $s = 2$ and $X_1 = \{x_1, x_2\}$. Since

$d(y_1) = d(y_2) = 2$ and $\sigma_2(G) \geq n-1$, $X = X_1$ is skew-cut of G with $A = \{y_1, y_2\}$ and $B = V(P) - X$, a contradiction. □

Let $y_s \in Y_1 \subseteq V(H)$ such that $x_s y_s \in E$.

Claim 3 *$d_P(y_1) = 1$ and $d_P(y_s) = 1$.*

Proof: Suppose, to the contrary and without loss of generality, $d_P(y_1) > 1$. Let

$$N_P(y_1) = \{x_{j_1}(= x_1), x_{j_2}, \ldots, x_{j_k}\},$$

where $j_1 < j_2 < \cdots < j_k$ and $k \geq 2$. Let $z_i = x_{j_i}^-$ and $w_i = x_{j_i}^+$ for each $i = 1, 2, \ldots, k$. Since P is a maximal path, $\{y_1\} \cup N_P^-(y_1)$ and $\{y_1\} \cup N_P^+(y_1)$ are independent vertex sets of G.

Since $z_1 = x_1^-$, we restate some properties of x_1^- as follows.

$$d(z_1) + d(y_1) = n-1, \ N(y_1) \supseteq V(H) - \{y_1\}, \ \text{ and } N(z_1) = V(P) - N_P^-(y_1). \quad (1)$$

In particular, we have that $z_1 v, z_1 v^- \in E$.

If $uz_i \in E$ for some $i \geq 2$, then $P^-[z_1, u]P^-[z_i, x_1]y_1 P[x_{j_i}, v]$ is a maximal path of order $p+1$, a contradiction. If $z_i v \in E$ for some $i \geq 2$, then $P[u, z_1]P^-[v^-, x_{j_i}]y_1 P[x_1, z_i]v$ is a maximal path of order $p+1$, a contradiction. Thus, $z_i u \notin E$ and $z_i v \notin E$ for each $i \geq 2$.

For each $i \geq 2$, since $z_i \notin N(y_1)$ and $N(z_1) = V(P) - N_P^-(y_1)$, $z_1 z_i^- \in E$. Let $Q = P[u, z_1]P^-[z_i^-, x_1]y_1 P[x_{j_i}, v]$. Since $z_i v \notin E$ and $z_i u \notin E$, Q is a maximal path of order p. We claim that $N(z_i) \cap V(P[u, z_1]) = \emptyset$. Otherwise, let $w \in V(P[u, z_1])$ such that $wz_i \in E$. Since $z_i u \notin E$, $w \neq u$. This together with $N(z_1) = V(P) - N_P^-(y_1)$ implies $w^- z_1 \in E$. Hence, $P[u, w^-]P^-[z_1, w]P^-[z_i, x_1]y_1 P[x_{j_i}, v]$ is a maximal path of order $p+1$, a contradiction. Thus, z_i^- is the first vertex along Q such that $N(z_i) - V(Q) \neq \emptyset$. Then z_i plays the same role in $G - V(Q)$ as y_1 does in H, so that $N(z_i) \supset V(H) - \{y_1\}$. Therefore, $N(y_s) \supset \{z_2, z_3, \cdots, z_k\}$.

Since $z_i y_1 \notin E$ and $z_i y_s \in E$ for each $i \geq 2$, we have $y_s \neq y_1$. Since $z_i y_s \in E$, $x_{j_i} y_s \notin E$ for each $i = 1, 2, \cdots, k$. Since x_s is the last vertex along $P[u, v]$ such that $N(x_s) - V(P) \neq \emptyset$, $x_s \in P[w_k, v]$.

Note that $N(y_s) \supset \{z_2, z_3, \cdots, z_k, x_s\}$. Similarly, considering path $P^-[v, u]$ and y_s, we have $N_H(x_{j_i}) = V(H) - \{y_s\}$ for each $i \geq 2$. Recall $N_H(z_i) = V(H) - \{y_1\}$. Thus, $N_H(x_{j_i}) \cap N_H(z_i) = V(H) - \{y_1, y_s\}$. Since $h = |V(H)| \geq 3$, there exists a vertex $y^* \in V(H)$ such that $y^* x_{j_i} \in E$ and $y^* z_i \in E$ for each $i \geq 2$. Then, $P[u, z_2]y^* P[x_{j_2}, v]$ is a maximal path of order $p+1$, a contradiction. □

Claim 4 *$Y_2 = \emptyset$, i.e. $d_P(y) \leq 1$ for each $y \in V(H)$.*

Proof: Suppose, to the contrary, $Y_2 \neq \emptyset$. Let $i_2 = \min\{i : x_i \in N_P(Y_2)\}$, and let $y \in Y_2$ such that $yx_{i_2} \in E$ and $yx_{i_3} \in E$, where $i_3 \neq i_2$. By Claim 3, $1 < i_2 < i_3 < s$. Let

$z_1 = x_1^-$, $z_2 = x_{i_2}^-$, $z_3 = x_{i_3}^-$, and $z_s = x_s^-$. Moreover, Let $P_1 = P[u, z_2]$, $P_2 = P[z_2^+, z_3]$, and $P_3 = P[z_3^+, v]$.

Since there does not exist a maximal path of order $p + 1$, we have the following equalities.

$$N_{P_1}^+(z_2) \cap N_{P_1}(z_3) = N_{P_2}(z_2) \cap N_{P_2}^+(z_3) = N_{P_3}(z_2) \cap N_{P_3}^-(z_3) = \emptyset.$$

Note that $N_{P_1}^+(z_2) \subseteq V(P_1)$, $N_{P_2}^+(z_3) \subseteq V(P_2)$, and $N_{P_3}^-(z_3) \subseteq V(P_3) \cup \{z_3\}$.

$$\begin{aligned}
& d_P(z_2) + d_P(z_3) \\
= \ & |N_{P_1}^+(z_2)| + |N_{P_2}(z_2)| + |N_{P_3}(z_2)| + |N_{P_1}(z_3)| + |N_{P_2}^+(z_3)| + |N_{P_3}^-(z_3)| \\
\leq \ & |(N^+(z_2) \cup N(z_3)) \cap V(P_1)| + |(N(z_2) \cup N^+(z_3)) \cap V(P_2)| + \\
& + |(N(z_2) \cup N^-(z_3)) \cap V(P_3)| + |\{z_3\}| \\
\leq \ & |V(P)| + 1 = p + 1
\end{aligned}$$

From the definition of x_{i_2}, we have $z_2 \notin X_2$, so that $(N(z_2) \cap N(z_3)) \cap V(H) = \emptyset$. Since $d(z_2) + d(z_3) \geq n - 1$, we have $d_H(z_2) + d_H(z_3) \geq h - 2$. Combining this inequality and the fact that y, $y_s \notin N(z_2) \cup N(z_3)$, we obtain that $N(z_2) \cup N(z_3) \supseteq V(H) - \{y, y_s\}$. Thus, $y_1 \in N(z_2) \cup N(z_3)$. Since $d_P(y_1) = 1$, $z_2 = x_1$. Moreover, we have $N(x_s) \cap V(H) = \{y_s\}$. Similarly, $N(x_1) \cap V(H) = \{y_1\}$.

For each $w \in V(P) - X_1$, since $d(y_1) + d(w) \geq n - 1$ and $d_P(y_1) = 1$, $N[w] = V(P)$. So $N[u] = N[z_1] = N[v] = V(P)$. In particular, $z_1 z_3 \in E$ and $v x_1 \in E$.

If $x_1 v^- \in E$, then $P[u, z_1]P^-[z_3, x_{i_2}]yP[x_{i_3}, v^-]x_1 v$ is a maximal path of order $p+1$, a contradiction. Thus, $x_1 v^- \notin E$. Since $d(x_1) + d(y_s) \geq n - 1$ and $d_P(y_s) = 1 = d_H(x_1)$, we have $d_P(x_1) \geq |V(P)| - 2$, so that $x_1 x_{i_2}^+ \in E$. Then $P[u, z_1]P^-[z_3, x_{i_2}^+]x_1 x_{i_2} y P[x_{i_3}, v]$ is a maximal path of order $p + 1$, a contradiction. □

Since each vertex in H has at most one neighbor on $P[u, v]$, we have $d_P(y) \leq 1$ for each $y \in V(H)$. On the other hand, since $d(u) + d(y) \geq n - 1$ and $N_H(u) = \emptyset$, we have $d_P(y) \geq 1$. Thus, $d_P(y) = 1$ for each $y \in V(H)$. Applying $\sigma_2(G) \geq n - 1$ again, we get both $V(H)$ and $V(P) - N_P(H)$ are cliques, and $N[w] = V(P)$ for each $w \in V(P) - X_1$. Then, X_1 is a skew-cut of G and G is a skew-joint graph, a contradiction. □

3 Proof of Theorem 4

3.1 A Lemma

Let G be a traceable graph and let $B \subseteq V(G)$. It is not difficult to see that $\omega(G - B) \leq |B| + 1$, where $\omega(G - B)$ is the number of components of $G - B$. We call $B \subseteq V(G)$ a *cutter* if $\omega(G - B) = |B| + 1$. Note that $\emptyset$ is a cutter of G by the definition. A cutter B is *maximum* if $|A| \leq |B|$ for every cutter A. A hamiltonian path P of G is called an *x-H-path* if x is one of two endvertices of P. If G is hamiltonian, then G contains an x-H-path for each $x \in V(G)$.

Lemma 1 *Let G be a P_4-free traceable graph and B be a maximum cutter of G. Then, for every $x \notin B$*

(1) G contains an x-H-path, and

(2) if $B \neq \emptyset$, then for any hamiltonian path $P[u,v]$, G contains an x-H-path such that the other end-vertex is either u or v.

Proof: If $B = \emptyset$, then G is 1-tough. By Theorem 3, G is hamiltonian. Then, G contains an x-H-path. So we assume $B \neq \emptyset$.

Let $P[u,v]$ be a hamiltonian path of G and let $B = \{b_1, b_2, \ldots, b_s\}$. Assume that b_1, b_2, $\ldots$, b_s are listed in the order along the orientation of $P[u,v]$. Since $\omega(G-B) = |B|+1$, B does not contain two consecutive vertices of $P[u,v]$, $u \notin B$, and $v \notin B$. Let $A_0 = V(P[u,b_1)]$, $A_i = V(P(b_i, b_{i+1}))$ for each $i = 1$, 2, $\ldots$, $s-1$, and $A_s = V(P(b_s, v])$. Since $\omega(G-B) = |B|+1$, $G_i := G[A_i]$ is a component of $G-B$ for each $i = 0$, $\ldots$, s. So, $E(A_i, A_j) = \emptyset$, for all $0 \leq i \neq j \leq s$, where $E(A_i, A_j) = \{ab \ : \ a \in A_i, \ b \in A_j\}$. Assume $x \in A_{i_0}$, where $0 \leq i_0 \leq s$. Since B is a maximum cutter, $G[A_{i_0}]$ is 1-tough. By Theorem 3, $G[A_{i_0}]$ contains an x-H-path $Q[x,y]$.

We claim that $N(b_i) \supseteq A_j$ if $N_{A_j}(b_i) \neq \emptyset$ for each pair i and j. Otherwise, since $G[A_j]$ is connected, there exist u_j, $v_j \in A_j$ such that $u_jv_j \in E$ and $b_iu_j \in E$ and $b_iv_j \notin E$. Then, either $b_i^- b_i u_j v_j$ (if $i \leq j$) or path $b_i^+ b_i u_j v_j$ (if $i > j$) is an induced P_4, a contradiction. In particular, we have $N(b_i) \supseteq A_i \cup A_{i-1}$ for each $1 \leq i \leq s$.

If $i_0 = 0$ then $Q[x,y]P[b_1, v]$ is a hamiltonian path of G. If $i_0 = s$ then $Q[x,y]P^-[b_s, u]$ is a hamiltonian path. We assume $0 < i_0 < s$.

We will recursively define a sequence of vertices b_{i_0}, b_{i_1}, $\cdots$, $b_{i_t} \in B$ which will help us to find an x-H-path. If $N(b_{i_0}) \supseteq \cup_{i \geq i_0} A_i$, let $t = 0$ and stop. Otherwise, let i_1 be the smallest i such that $N_{A_i}(b_{i_0}) = \emptyset$. Since G is P_4-free, $N_{A_{i_0-1}}(b_{i_1}) \neq \emptyset$. If $N(b_{i_1}) \supseteq \cup_{i<i_0} A_i$, let $t = 1$ and stop. Otherwise, let A_{i_2} be the largest $i < i_0$ such that $N_{A_{i-1}}(b_{i_1}) = \emptyset$. Since G is P_4-free, $N_{A_{i_1+1}}(b_2) \neq \emptyset$. If $N(b_{i_2}) \supseteq \cup_{i \geq i_1} A_i$, let $t = 2$ and stop. Otherwise, let i_3 be the smallest i such that $N_{A_i}(b_{i_2}) = \emptyset$. Since G is P_4-free, $N_{A_{i_2-1}}(b_3) \neq \emptyset$. Continuing in this manner, since G is finite, we obtain a finite sequence i_0, i_1, i_2, $\cdots$, i_t with $i_0 < i_1 < i_3 < \cdots < i_{2\ell+1} < \cdots$ and $i_0 > i_2 > i_4 > \cdots > i_{2\ell} > \cdots$ such that

$$\begin{array}{rcl}
N(b_{i_0}) & \supseteq & A_{i_0} \cup A_{i_0+1} \cup \cdots \cup A_{i_1-1} \\
N(b_{i_1}) & \supseteq & A_{i_0-1} \cup A_{i_0-2} \cup \cdots \cup A_{i_2} \\
N(b_{i_2}) & \supseteq & A_{i_1} \cup A_{i_1+1} \cup \cdots \cup A_{i_3-1} \\
N(b_{i_3}) & \supseteq & A_{i_2-1} \cup A_{i_2-2} \cup \cdots \cup A_{i_4} \\
\vdots & \vdots & \vdots
\end{array}$$

and

$$N(b_{i_t}) \supseteq \begin{cases} \cup_{i \geq i_{t-1}} A_i & \text{if } t \text{ is even,} \\ \cup_{i < i_{t-1}} A_i & \text{if } t \text{ is odd.} \end{cases}$$

If $t = 0$, $Q[x,y]P[b_{i_0+1}, v]P^-[b_{i_0}, u]$ is a hamiltonian path from x to u in G.

If $t > 0$ is even, G contains a hamiltonian path from x to u as follows.

$$Q[x,y]P[b_{i_0+1}, b_{i_1}^-]P^-[b_{i_0}, b_{i_2}^+]P[b_{i_1}, b_{i_3}^-] \cdots P[b_{i_{t-1}}, v]P^-[b_{i_t}, u].$$

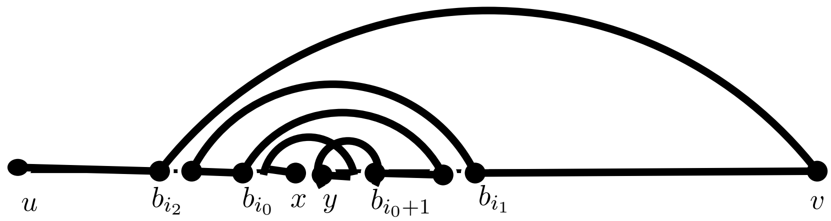

Figure 4: The case of $t = 2$

The case $t = 2$ is illustrated in Figure 4.

If t is odd, G contains a hamiltonian path from x to v as shown below:

$$Q[x, y]P[b_{i_0+1}, b_{i_1}^-]P^-[b_{i_0}, b_{i_2}^+] \cdots P^-[b_{i_{t-1}}, u]P[b_{i_t}, v].$$

The case $t = 3$ is illustrated in Figure 5. So, in each case, G has an x-H-path. □

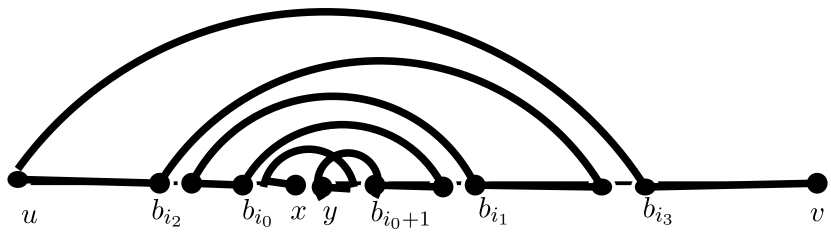

Figure 5: The case of $t = 3$

3.2 Proof of Theorem 4

Suppose, to the contrary, there is a P_4-free and 1-tough graph G such that G contains a maximal path $P = P[u, v]$ of order p and G does not contain a maximal path of order $p + 1$, for some $p \leq n - 2$. Let H be a component in $G - V(P[u, v])$.

Claim 5 *For each $x \in V(P[u, v])$, either $N_H(x) = \emptyset$ or $N_H(x) = V(H)$.*

Proof: Suppose, to the contrary, there is a vertex $x \in V(P)$ such that $\emptyset \neq N_H(x) \neq V(H)$. Clearly, $x \neq u$. Further, we assume that x is the one closest to u on P with the above property. Since H is connected, there are $y,\ z \in V(H)$ such that $yz \in E$, $xy \in E$, and $xz \notin E$. Since G does not contain a maximal path of order $p + 1$, $x^-y \notin E$. By our choice of x, we have $N_H(x^-) = \emptyset$. So, x^-xyz is an induce P_4, a contradiction. □

Claim 6 *$N_P(H)$ does not contain two consecutive vertices of P for each component H of $G - V(P)$.*

Proof: Suppose, to the contrary, there are two consecutive vertices $w, x \in V(P)$ such that $N_H(w) \neq \emptyset$ and $N_H(x) \neq \emptyset$. By Claim 5, $N(w) \cap N(x) \supset V(H)$. Then, $P[u, w]yP[x, v]$ is a maximal path of order $p + 1$, where $y \in V(H)$, a contradiction. □

Since G is P_4-free, Claim 6 implies the following claim.

Claim 7 *$N_P(G-V(P))$ does not contain two consecutive vertices of P for each maximal path P of order p.*

Let $N_P(H) = \{x_1, x_2 \ldots, x_s\}$, where x_1, x_2, …, x_s are listed in the order along the orientation of $P[u,v]$. Since G is 1-tough, G is 2-connected, so that $s \geq 2$. Since $P[u,v]$ is a maximal path, $x_1 \neq u$ and $x_s \neq v$. Let

$$\begin{aligned} A_0 &= V(P[u,x_1)) \quad \text{and} \quad G_0 = G[A_0], \\ A_i &= V(P(x_i,x_{i+1})) \quad \text{and} \quad G_i = G[A_i] \quad \text{for } i = 1,\, 2,\, \cdots, s-1, \\ A_s &= V(P(x_s,v]) \quad \text{and} \quad G_s = G[A_s]. \end{aligned}$$

Let y denote an arbitrary vertex of H in the remainder of the proof. Since G does not contain an induced P_4, the following results hold.

Claim 8 *For any two integers $i = 1, 2, \ldots, s$ and $j = 0, 1, \ldots, s$,*

1. *$N(x_i) \supset A_{i-1}$ and $N(x_i) \supset A_i$, and*
2. *$N(x_i) \supseteq A_j$ if $N_{A_j}(x_i) \neq \emptyset$.*

For each G_i, let B_i be a maximum cutter of G_i and $C_i = A_i - B_i$. Note that $B_i = \emptyset$ may happen. By the definition of cutter, $G_i - B_i$ contains exactly $|B_i| + 1$ components.

Claim 9 *For each $v_i \in C_i$, there exists a vertex $w_i \in A_i$ such that G_i contains a hamiltonian path from v_i to w_i and $N(v_i) \cup N(w_i) \subseteq V(P[u,v])$.*

Proof: By Lemma 1, G_i contains a v_i-H-path $Q[v_i,w_i]$. We only need to show that $N(v_i) \cup N(w_i) \subseteq V(P[u,v])$. Suppose, to the contrary, there exists an integer i such that $N(v_i) \cup N(w_i) \not\subseteq V(P[u,v])$.

If $i \neq 0$ and $i \neq s$, $P^* = P[u,x_i]Q[v_i,w_i]P[x_{i+1},v]$ is a maximal path of order p. By Claim 7, $N_{P^*}(G - V(P))$ does not contain two consecutive vertices of P^*. Since $x_i, x_{i+1} \in N_{P^*}(H)$, we have $N(v_i) \subseteq V(P)$ and $N(w_i) \subseteq V(P)$, a contradiction. Thus, we may assume either $i = 0$ or $i = s$, say, without loss of generality, $i = 0$.

If G_0 is not 1-tough, by Lemma 1, we may assume that $w_0 \in \{u, x_1^-\}$. Since P is a maximal path, $N(u) \subseteq V(P)$. Since $N(x_1) \cap V(H) \neq \emptyset$, $N(x_1^-) \subseteq V(P)$ by Claim 7. Thus, $N(w_0) \subseteq V(P)$ regardless $w_0 = u$ or $w_0 = x_1^-$. Applying Claim 7 to the maximal path $Q_0^-[w_0,v_0]P[x_1,v]$, we have $N(v_0) \subseteq V(P[u,v])$, a contradiction. Thus, G_0 is 1-tough.

By Theorem 3, G_0 contains a hamiltonian cycle C. For each $x \in V(C)$, let $x^-(C)$ denote the predecessor of x on C. Then, $P' = C[u,u^-(C)]P[x_1,v]$ is a maximal path. Note that $V(P) = V(P')$. By Claim 7, either $N(v_0) \subseteq V(P)$ or $N(v_0^-(C)) \subseteq V(P)$. Suppose $N(v_0^-(c)) \subseteq V(P)$. Applying Claim 7 to maximal path $C^-[v_0^-(C),v_0]P[x_1,v]$, we have $N(v_0) \subseteq V(P)$. which implies Claim 9 is true with $w_0 = v_0^-(C)$. □

Claim 10 *$E(C_i,C_j) = \emptyset$ for each $0 \leq i < j \leq s$.*

Proof: Suppose, to the contrary, there exist $v_i \in C_i$ and $v_j \in C_j$ with $0 \le i < j \le s$ such that $v_iv_j \in E$. For each $\ell = i, j$, by Claim 9, there exists a hamiltonian path $Q_\ell[v_\ell, w_\ell]$ such that $N(v_\ell) \cup N(w_\ell) \subseteq V(P[u, v])$. Since $x_1 \in N_P(H)$, $N(x_1^+) \subseteq V(P[u, v])$ by Claim 7. Let y be an arbitrary vertex of H. By Claim 5, $N_P(y) = N_P(H) = \{x_1, x_2 \ldots, x_s\}$. Set

$$P' = \begin{cases} P[u, x_i]yP^-[x_j, x_{i+1}]Q_i^-[w_i, v_i]Q_j[v_j, w_j]P[x_{j+1}, v], & \text{if } i \neq 0 \text{ and } j \neq s \\ Q_0^-[w_0, v_0]Q_j[v_j, w_j]P^-[x_j, x_1]yP[x_{j+1}, v], & \text{if } i = 0 \text{ and } j \neq s \\ P[u, x_i]yP^-[x_s, x_{i+1}]Q_i^-[w_i, v_i]Q_s[v_s, w_s], & \text{if } i \neq 0 \text{ and } j = s \\ P[x_1^+, x_s]yx_1Q_0^-[w_0, v_0]Q_s[v_s, w_s], & \text{if } i = 0 \text{ and } j = s. \end{cases}$$

Clearly, P' is a maximal path of order $p + 1$, a contradiction. $\Box$

Let $S = \{x_1, x_2, \ldots, x_k\} \cup B_0 \cup B_1 \cup \cdots \cup B_k$, where B_i is a maximum cutter of G_i for each $i = 0,\ 1,\ \cdots,\ s$. By Claim 9, we have $N(V(P[u, v] - S)) \subseteq V(P[u, v])$. This together with Claim 10 implies that every segment of $P[u, v] - S$ induces a connected component in $G - S$. These components and the component H show that $G - S$ has at least $|S| + 2$ components, which contradicts the assumption that G is 1-tough. This completes the proof of Theorem 4. $\Box$

4 Proof of Theorem 6

Suppose, to the contrary, there exists a connected, locally connected, and $K_{1,3}$-free graph G such that G contains a maximal path $P = P[u, v]$ of order p and G does not contain a maximal path of order $p+1$. Since G is traceable, $p \le n-2$. Let $H = G-V(P)$ and let y be a vertex of H with $N_P(y) \neq \emptyset$. Let $N_P(y) = \{x_1,\ x_2\ \cdots, x_s\}$, where $x_1,\ x_2,\ \ldots,\ x_s$ are listed in the order along the orientation of $P[u, v]$. Choose y so that $\min\{|P[u, x_1)|,\ |P[x_s, v]|\}$ achieves the minimum. Without loss of generality, assume $|P[u, x_1]| \le |P[x_s, v]|$. Since $P[u, v]$ is a maximal path, $x_1 \neq u$, $x_s \neq v$ and $\left(\{x_1^-,\ x_2^-,\ \cdots,\ x_s^-\} \cup \{x_1^+,\ x_2^+,\ \cdots,\ x_s^+\}\right) \cap N_P(y) = \emptyset$.

Since G is $K_{1,3}$-free, $x_i^- x_i^+ \in E(G)$ for all $i = 1,\ 2,\ \cdots,\ s$. Let $x = x_1$, $A = V(P) - N_P(y)$, and $B = V(H) \cup N_P(y)$.

Claim 11 *$N_H(x^-) = \emptyset$ and $N_P(x^-) \cap N_P(y) = \{x\}$.*

Proof: $N_H(x^-) = \emptyset$ directly comes from the definition of P and y. If $x^-x_i \in E$ for some $i > 1$, $P[u, x^-]x_iyP[x, x_i^-]P[x_i^+, v]$ is a maximal path of order $p + 1$, a contradiction. $\Box$

Claim 12 *Both $G[N_A(x)]$ and $G[N_B(x)]$ are complete.*

Proof: Since $G[N_A[x] \cup \{y\}]$ contains no induced $K_{1,3}$, we have $G[N_A(x)]$ is complete. Since $G[N_B[x] \cup \{x^-\}]$ contains no induced $K_{1,3}$, $G[N_B(x)]$ is complete. $\Box$

Since G is locally connected, there exist $a \in N_A(x)$ and $b \in N_B(x)$ such that $ab \in E$.

Claim 13 *$b \notin V(H)$.*

Proof: Suppose, to the contrary, that $b \in V(H)$. Then, $a \in N_P(H)$. Since G is $K_{1,3}$-free, $a^-a^+ \in E$. By the choice of y, we have $a \in V(P[x^+, v])$. Since $x^-, a \in N_A(x)$, $x^-a \in E(G)$ from Claim 12. Then, $P[u, x^-]abP[x, a^-]P[a^+, v]$ is a maximal path of order $p + 1$, a contradiction. □

Assume $b = x_i$ for some $i \geq 2$. Since $x^-x^+ \in E$ and $x_i^-x_i^+ \in E$, we have $x_i^-, x_i^+ \notin N(x)$ and $x^-, x^+ \notin N(x_i)$. Otherwise, for example, $xx_i^- \in E$, then $P[u, x^-]P[x^+, x_i^-]xyP[x_i, v]$ id s maximal path of order $p + 1$, a contradiction. Since $ax \in E$ and $ab = ax_i \in E$, $a \notin \{x^-, x^+, x_i^-, x_i^+\}$. Since $a, x^- \in N_A(x)$, $ax^- \in E$ by Claim 12. We will consider the following two cases to finish the proof.

Case 1 $a \neq u, v$.

Since $G[\{a, a^-, a^+, x\}]$ is not an induced $K_{1,3}$, $\{a^-a^+, a^-x, a^+x\} \cap E \neq \emptyset$. We will derive a contradiction by showing that G has a $u - v$ path P' of order $p+1$ with $V(P') \supseteq V(P)$, which is equivalent to that $G \cup \{uv\}$ contains a cycle C' of order $p + 1$ with $uv \in E(C')$ and $V(C') \supseteq V(P)$. Set $C = P[u, v]u$. Then, C is a cycle of order p in $G \cup \{uv\}$. Then, either $a \in C(x^+, x_i^-)$ or $a \in C(x_i^+, x^-)$. Assume, without loss of generality, that $a \in C(x^+, x_i^-)$ (the case of $a \in C(x_i^+, x^-)$ is similar). Let

$$C' = \begin{cases} x^-ax_iyC[x, a^-]C[a^+, x_i^-]C[x_i^+, x^-] & \text{if } a^-a^+ \in E, \\ x^-C[x^+, a^-]xyx_iC[a, x_i^-]C[x_i^+, x^-] & \text{if } a^-x \in E, \\ x^-C[x^+, a]x_iyxC[a^+, x_i^-]C[x_i^+, x^-] & \text{if } a^+x \in E. \end{cases}$$

In each case, C' is a cycle in $G \cup \{uv\}$ with $uv \in E(C')$ and $V(C') = V(C) \cup \{y\}$. So, $C' - uv$ is a maximal path of order $p + 1$, a contradiction.

Case 2 $a \in \{u, v\}$.

Recall $x = x_1$. From the minimality of $|P[u, x_1]|$, we obtain that $N(x^-) \subseteq V(P)$. If $a = u$, then $P^-[x^-, u]x_iyP[x, x_i^-]P[x_i^+, v]$ is a maximal path of order $p+1$, a contradiction. Hence, $a = v$.

Recall $x^-a \in E(G)$. If $N(v^-) \subseteq V(P)$, then $P[u, x^-]vx_iyP[x, x_i^-]P[x_i^+, v^-]$ is a maximal path of order $p + 1$. Hence, $N(v^-) \not\subseteq V(P)$. From the minimality of $|C[u, x_1]|$, we have $|P[u, x]| \leq |P[v^-, v]|$. So, $u = x^-$. This together with Claim 11 implies $ux_i \notin E$. Since $u, v \in N_A(x)$, by Claim 12, we have $uv \in E$. Note that $a \neq x_i^+$ implies that $x_i \neq v^-$. Hence, v, u, x_i and v^- are distinct vertices of G. Since $G[\{v, u, x_i, v^-\}]$ is not an induced $K_{1,3}$ and $ux_i \notin E$, either $uv^- \in E$ or $x_iv^- \in E$. Let

$$P' = \begin{cases} uP^-[v^-, x_i^+]P^-[x_i^-, x]yx_iv, & \text{if } uv^- \in E, \\ uP[x^+, x_i^-]P[x_i^+, v^-]x_iyxv, & \text{if } x_iv^- \in E. \end{cases}$$

Then, P' is a maximal path of order $p + 1$, a contradiction. This completes the proof of Theorem 6. □

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