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COVERING THE VERTICES OF A GRAPH BY CYCLES OF BOUNDED LENGTH

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ABSTRACT. Let $c_k(G)$ be the minimum number of elementary cycles of length at most k necessary to cover the vertices of a given graph G. In this work, we bound $c_k(G)$ in function of the order of the graph and its independence number.

Key words: Cycle; Vertex Covering; Independence Number.

1. INTRODUCTION

Throughout this paper, we consider finite simple graphs $G = (V, E)$ and we denote by n the order of the considered graph.
The distance between two vertices u and v in G, is denoted by $d_G(u, v)$, and is defined to be the length of a shortest path joining them in G.
The size of a largest independent set of G is called the independence number of G and is denoted by $\alpha(G)$ when no ambiguity.
A *covering* of a graph G is a family of elementary cycles of G such that each vertex of G lies in at least one cycle of this family. For terms not defined here, we refer to [1].
In the literature, many results dealing with the covering of a graph by cycles have appeared. Corrdi and Hajnal (in [3]) have proved a result conjectured a few years before by Erds, which is that if G is a graph of order $n \geq 3k$ with minimum degree $\delta \geq 2k$, then G contains k vertex disjoint cycles. Later on, several authors have been, in some sense, inspired by this theorem and have sharpened it in many ways. In [9], Lesniak has discussed a variety of results dealing with the existence of disjoint cycles in a given graph.
In [5] and [10], Enomoto and Wang have relaxed the degree condition given by Erds. They have independently established that a graph of order at least $3k$ in which $d(u)+d(v) \geq 4k-1$ for every pair of non adjacent vertices u and v, contains k vertex disjoint cycles. In [4], Egawa and al. have showed that by taking three integers d, k, and n such that $k \geq 3, d \geq 4k-1$ and $n \geq 3k$ and a graph G of order n, in which each pair of nonadjacent vertices x and y verifies $d(x) + d(y) \geq d$, then at least $\min(d, n)$ vertices of G can be covered by k vertex disjoint cycles.
However, in what precedes, the interest was given to the independence of the cycles rather than the fact that they cover all the vertices of the graph. In [7], Kouider and Lonc have showed that the vertices of a 2-connected graph in which $\displaystyle\sum_{x \in S} d_G(x) \geq n$ for every independent set S of cardinality s can be covered by at most $s-1$ cycles. In another paper[8], Kouider

shows that the vertices of any κ connected graph are covered by at most $\lceil \alpha/\kappa \rceil$ cycles.
But in all these results, no bound for the length of the cycles taken in the covering is imposed. Recently, in [6], Forge and Kouider have laid down that the cycles taken in the covering are of length not exceeding k (where k is an integer preliminary fixed). They have denoted by $c_k(G)$ the cardinality of a minimum covering in which each cycle satisfies the previous condition. They have bounded $c_k(G)$ in function of the minimum degree and the order of the graph G. They have also introduced the concept of *alternated* cycles to show that:
If p and k are two integers such that $2 \leq p \leq \frac{k}{8}$ and if G is a graph of order $n \geq \frac{2k}{3}(p-1)^2 + (p-1)$ and minimum degree δ at least $\frac{n}{p} + \frac{2k}{3}$. Then:

$$c_k(G) \leq \frac{3n}{k} + \frac{\log \frac{k}{3}}{-\log(1 - \frac{1}{2(p-1)^2})} + (1 - \frac{3}{k})(p-2) + 1.$$

In this work, we intend to bound $c_k(G)$ in function of the independence number of the graph and its order and we show among others the following result:

Theorem 1.1. *Let G be a 2-connected graph of order n with stability number α and k an integer such that* $\frac{(k+1)}{2(\alpha+1)} \geq 2$. *Then* $c_k(G) \leq \frac{n}{k - \frac{4}{3}(\alpha+1)} + \alpha \log k$ *if* $n > \alpha(k - \frac{4}{3}(\alpha+1))$.

2. Covering the vertices by cycles of length at most k

Let k be an integer and G a graph of order n. We want to cover G by the smallest possible number of cycles of length at most k.
Each time we have a cycle in G, we check its length, if it is less than or equal to k then this cycle is taken in the covering otherwise, a chord may reduce its length. Therefore, we should assume that $k \geq 2\alpha + 1$ so that there always exists a chord (at least).
In what follows, we show that according to the prescribed value of k we can guarantee the existence in G of a cycle of length not only at most k but at least a fraction of k as well.

Proposition 2.1. *Let G be a graph of order n and independence number α and let k be an integer such that $k \geq 2\alpha + 1$. If G has a cycle of length more than k, then it has a cycle of length between* $\frac{k+1}{2}$ *and k.*

Proof. Indeed, if C is a cycle of G of length $l(C)$ at least $k+1 \geq 2\alpha + 2$, then there are at least $\alpha + 1$ independent vertices on C and thus at least two of these vertices (say x and y) are adjacent. Furthermore, $2 \leq d_C(x, y) \leq \frac{l(C)}{2}$. The chord (x, y) divides the cycle C into two smaller cycles, the bigger C_1 is length $l(C_1)$ between $\frac{l(C)}{2}$ and $l(C) - 1$. We resume the same construction until we get a cycle C_i such that $\frac{k+1}{2} \leq l(C_i) \leq k$. □

If k is supposed greater than what it is in the previous theorem then the length of the obtained cycle will be bounded on the left by a bigger fraction of k.

Proposition 2.2. *Let G be a graph of order n with independence number α and let k be an integer such that $k \geq 4\alpha+3$. If G possesses a cycle of length at least $\frac{2k}{3}$, then it has a cycle of length between $\frac{2k}{3}$ and k.*

Proof. Let C be a cycle of G of length $l \geq \frac{2k}{3}$.
If $l \leq k$ then C is a cycle of length between $\frac{2k}{3}$ and k.
In the case where $l > k$, we are going to construct a cycle of length at least $\frac{2k}{3}$ and strictly smaller than l. Clearly by iterating the construction we will finally get a cyle of length between $\frac{2k}{3}$ and k.
Consider on the cycle an orientation O. We will use $d_O(x,y)$ to be the distance on the cycle using the orientation O. Consider among all the possible sets $\{v_1, ..., v_{\alpha+1}\}$ of $(\alpha+1)$ distinct vertices such that $d_O(v_i, v_{i+1}) = 2$ for $1 \leq i \leq \alpha$ the one that contains two joined vertices v_1 and v_i (joined in G) at minimum distance on C.

- If $d_O(v_1, v_i) \leq \frac{l}{3}$ then we have the desired cycle.
- If not, then consider the following set $S = \{v_2, ...v_{\alpha+1}, v_{\alpha+2}\}$ where $d_O(v_{\alpha+1}, v_{\alpha+2})$ is also 2 on C. Let v_j and v_r be two joined vertices of S(as $|S| = \alpha+1$). We cannot have $j \geq i$ otherwise, since $d_O(v_j, v_r) \geq d_O(v_1, v_i) > \frac{l}{3}$ then $d_O(v_1, v_{\alpha+2}) \geq d_O(v_1, v_i) + d_O(v_j, v_r) \geq \frac{2l}{3}$ but $d_O(v_1, v_{\alpha+2}) \leq \frac{l}{2}$ (because $l \geq 4(\alpha+1)$). We get $\frac{l}{2} \geq \frac{2l}{3}$ which is a contradiction. Thus the segments $[v_1, v_i]$ and $[v_j, v_r]$ of C do intersect in at least two vertices. Let $l_1 = d_O(v_1, v_j)$, $l_2 = d_O(v_j, v_i)$ and $l_3 = d_O(v_i, v_r)$. We have: $l_1 + l_2 + l_3 \leq \frac{l}{2}$ and $l_1 + 2l_2 + l_3 \geq \frac{2l}{3}$. It follows that $l_2 \geq \frac{l}{6}$ and consequently the cycle $C' = (v_1, v_i)\bigcup[v_i, v_j]\bigcup(v_j, v_r)\bigcup[v_r, v_1]$ is of length $l' \geq \frac{2l}{3}$. This completes the proof.

□

More generally, for an integer $c \geq 2$ and for $k \geq 2c(\alpha+1)-1$ we have the following result:

Proposition 2.3. *Let G be a graph of order n with independence number α. Let c and k be two integers such that $c \geq 2$ and $k \geq 2c(\alpha+1)-1$. If G possesses a cycle of length at least $(1-\frac{2}{3c})k$, then it has a cycle of length between $(1-\frac{2}{3c})k$ and k.*

Proof. We use the definitions and techniques of the preceeding proof. Let C be a cycle of G of length $l \geq (1-\frac{2}{3c})k$.
If $l \leq k$ then C is as desired.
Otherwise, consider among all the possible sets $\{v_1, ..., v_{\alpha+1}\}$ of $(\alpha+1)$ vertices such that $d_O(v_i, v_{i+1}) = 2$ for $1 \leq i \leq \alpha$ the one that contains two joined vertices v_1 and v_i at minimum distance on C.

- If $d_O(v_1, v_i) \leq \frac{2l}{3c}$ then we have the desired cycle.
- If $d_O(v_1, v_i) > \frac{2l}{3c}$ then consider the following set $S = \{v_2, ...v_{\alpha+1}, v_{\alpha+2}\}$, where $d_O(v_{\alpha+1}, v_{\alpha+2})$ is also 2 on C. Let v_j and v_r be two joined vertices of S. We have $j < i$ otherwise, on one hand $d_O(v_j, v_r) \geq d_O(v_1, v_i) > \frac{2l}{3c}$ then $d_O(v_1, v_{\alpha+2}) \geq d_O(v_1, v_i) + d_O(v_j, v_r) \geq \frac{4l}{3c}$ and the other hand $d_O(v_1, v_{\alpha+2}) \leq \frac{l}{c}$ (since $l \geq 2c(\alpha+1)$). We get $\frac{4l}{3c} \leq \frac{l}{c}$ which is absurd. Thus the segments $[v_1, v_i]$ and $[v_j, v_r]$ of the cycle C do intersect in at least two vertices. Let $l_1 = d_O(v_1, v_j)$, $l_2 = d(v_j, v_i)$ and $l_3 = d_O(v_i, v_r)$. We have: $l_1 + l_2 + l_3 \leq \frac{l}{c}$ and $l_1 + 2l_2 + l_3 \geq \frac{4l}{3c}$. So $l_2 \geq \frac{l}{3c}$ and as a result the cycle $C' = (v_1, v_i) \bigcup [v_i, v_j] \bigcup (v_j, v_r) \bigcup [v_r, v_1]$ is of length $l' \geq (1 - \frac{2}{3c})l$ as desired.

$\square$

Remark 2.4. *Let G be a graph of order n with independence number α and let c and k be two integers, if we suppose that $c \geq 1$ and $k \geq 2c(\alpha + 1) - 1$ then the statements of Proposition 1 and Proposition 3 amount to the following: " G has a cycle of length between $\max((1 - \frac{2}{3c})k, \frac{k+1}{2})$ and k."*

In the previous propositions, we supposed that a cycle exists to begin the construction.The next propositions of [2] ensure the existence (maybe by adding conditions) of at least a cycle in G of sufficient length.

Proposition 2.5. *Let G be a connected graph of independence number α, then G possesses a cycle, an edge or a vertex which removal reduces its independence number of at least 1.*

Proof. Let P be a longest path in G and let x and y be its endpoints. All the neighbors of x and y are on P otherwise we get a contradiction. Two cases may occur:

(1) x and y are not of degree 1 in G, then we consider u the furthermost neighbor of x on P. The cycle C made of the segment $[x, u]$ on P and the edge (x, u) contains x and all its neighbors. Thus if we remove it, we get a graph with smaller independence number: $\alpha(G - C) \leq \alpha(G) - 1$.
(2) x or y is of degree 1 in G. Assume that it's x, then by suppressing the edge e of extremity x we get: $\alpha(G - e) \leq \alpha(G) - 1$.

$\square$

Proposition 2.6. *If G is a connected graph of independence number α then G can be covered by at most α disjoint cycles, edges or vertices.*

Proof. By induction on α

- If $\alpha = 1$ then the graph G is a clique and hence it can be covered by one cycle (a hamiltonian one).
- We may assume that every graph of independence number $\alpha - 1$ can be covered by at most $\alpha - 1$ disjoint cycles, edges or vertices and let G be a graph with independence number α. According to the previous theorem, there exists a cycle, an edge or a

vertex which removal gives a graph G' such that $\alpha(G') \leq \alpha(G) - 1 = \alpha - 1$. Then, G' can be covered by at most $\alpha - 1$ disjoint cycles, edges or vertices and it follows that G can be covered by at most α disjoint cycles, edges or vertices.

□

By combining all what have preceded, and by supposing moreover that G is 2-connected with a vertex set large enough and independence number at least 2, then we can cover G by at most a number of order $\frac{n}{(1-\frac{2}{3c})k}$ of cycles of length at most k, as states the following result:

Theorem 2.7. *Let G be a 2-connected graph of order n with independence number $\alpha > 1$. Let c and k be two integers such that $c \geq 2$ and $k \geq 2c(\alpha+1) - 1$. If $n \geq \alpha(1 - \frac{2}{3c})k$, then*

$$c_k(G) \leq \frac{n}{(1-\frac{2}{3c})k} + \lceil \frac{\log \frac{3}{(1-\frac{2}{3c})k}}{\log(1-\frac{1}{\alpha})} \rceil + \alpha.$$

Proof. Denote by N the set of uncovered vertices (at the beginning $|N| = n$).

Step 1 While $|N| \geq \alpha(1 - \frac{2}{3c})k$, then by Theorem 5, we have a cycle of length at least $\frac{|N|}{\alpha} \geq (1 - \frac{2}{3c})k$. If this cycle's length is greater than k then, by Theorem 3, we know how to reduce it, obtaining in any case a cycle which covers at least $(1 - \frac{2}{3c})k$ vertices of N. At the end of this step, at most $\frac{n - \alpha(1-\frac{2}{3c})k}{(1-\frac{2}{3c})k}$ cycles would be used.

Now $|N| < (1 - \frac{2}{3c})k\alpha$

Step 2 While $|N| \geq 3\alpha$, then by Theorem 5 we can find at least a cycle in N. The number of cycles used (which lengths are reduced by applying Theorem 3) is given by the (smallest) number i of iterations done until $|N|$ becomes $< 3\alpha$. After the first iteration, there remains at most $|N| - \frac{|N|}{\alpha} = |N|(1 - \frac{1}{\alpha})$ uncovered vertices. After i iterations, there lasts at most $|N|(1-\frac{1}{\alpha})^i$ uncovered vertices. We stop when $|N|(1-\frac{1}{\alpha})^i$ becomes $< 3\alpha$, and as far as that goes it suffices that $(1 - \frac{2}{3c})k\alpha(1-\frac{1}{\alpha})^i \leq 3\alpha$. It follows that $i \geq \lceil \frac{\log \frac{3}{(1-\frac{2}{3c})k}}{\log(1-\frac{1}{\alpha})} \rceil$.

When this step is over, we have $|N| < 3\alpha$.

Step 3 While $|N|$ is $> \alpha$, we can cover its vertices 2 by 2 (by Theorem 5) and since the considered graph G is 2-connected, then every edge lies in a cycle. If the length of this cycle is greater than k then we know how to reduce it (Theorem 3). Thus we obtain at most α further cycles in the covering.

And finally, when $|N| \leq \alpha$ we can cover the vertices one by one and for the same aforementioned reasons, we get at most α additional cycles in the covering. In short, we have a covering of G by a most: $\dfrac{n}{(1-\frac{2}{3c})k}+\dfrac{\log\frac{3}{(1-\frac{2}{3c})k}}{\log(1-\frac{1}{\alpha})}+\alpha+1$

Furthermore, $\log(1-\dfrac{1}{\alpha})<-\dfrac{1}{\alpha}$ (study the sign of the function $f(\alpha)=\log(1-\dfrac{1}{\alpha})+\dfrac{1}{\alpha}$) then $c_k(G)\leq\dfrac{n}{(1-\frac{2}{3c})k}+\alpha\log(\dfrac{(1-\frac{2}{3c})k}{3})+\alpha+1.$ □

Remark 2.8. (1) *In order that the function* $\log(1-\dfrac{1}{\alpha})$ *would be defined, the case* $\alpha=1$ *has been put aside. If this case occurs, then the 2-connected graph* G *would be a clique and hence it could be covered by at most* $\lceil\dfrac{n}{k}\rceil$ *cycles.*

(2) *More generally, by taking just a non zero integer* c*, the same bound holds by replacing* $(1-\dfrac{2}{3c})k$ *by* $\gamma=\max((1-\dfrac{2}{3c})k,\dfrac{k+1}{2})$*. Notice that the more* c *is great, the more* γ *approaches* k*.*

The previous bound remains for $c_k(G)$ even if n is not as large as supposed in the previous theorem. However, it can be improved as states the following result:

Theorem 2.9. *Let* G *be a 2-connected graph of order* n *with independence number* $\alpha>1$*. Let* c *and* k *be two integers such that* $c\geq 1$, $k\geq 2c(\alpha+1)-1$ *and* $\gamma=\max((1-\dfrac{2}{3c})k,\dfrac{k+1}{2})$*.*
If $n>\alpha\gamma$ *then* $c_k(G)\leq\dfrac{n}{\gamma}+\alpha(1+\log\dfrac{\gamma}{3})+1$*,*
if $3\alpha<n\leq\alpha\gamma$ *then* $c_k(G)\leq\alpha(2+\log\dfrac{\gamma}{3})+1$*,*
and if $n\leq 3\alpha$ *then* $c_k(G)\leq 2\alpha$*.*

Proof. The proof of the first case is already done.
The proofs of the two other cases are quite similar starting from step 2 and step 3 respectively in the previous proof. □

The first term of the given bound for $c_k(G)$ is fairly significant, since it is of order $\dfrac{3c\alpha}{(3c-2)}$ (where $c\geq 2$) for the graph which consists of α copies of K_k with a common vertex and which can be covered by α cycles.

For the complete graph K_n (n very large), we've got a covering by $\lceil\dfrac{n}{k}\rceil$ cycles, which is not so far from $\dfrac{n}{(1-\frac{2}{3c})k}+\log(\dfrac{(1-\frac{2}{3c})k}{3})+2$ given by the previous theorem for $k\geq 2c(\alpha+1)-1$ and c very large.

We deduce naturally the following corollaries from the preceding theorem.

Corollary 2.10. *Let* G *be a 2-connected graph of order* n *with independence number* $\alpha>1$*. Let* k *be an integer such that* $k\geq 2\alpha+1$*.*
If $n>\alpha(\dfrac{k+1}{2})$ *then* $c_k(G)\leq\dfrac{2n}{k+1}+\alpha(1+\log\dfrac{k+1}{6})+1$*,*

if $3\alpha < n \leq \alpha(\frac{k+1}{2})$ *then* $c_k(G) \leq \alpha(2+\log\frac{k+1}{6})+1$,
and if $n \leq 3\alpha$ *then* $c_k(G) \leq 2\alpha$.

Corollary 2.11. *Let G be a 2-connected graph of order n with stability number α and k an integer such that $\frac{(k+1)}{2(\alpha+1)} \geq 2$. Then*
$c_k(G) \leq \frac{n}{k-\frac{4}{3}(\alpha+1)} + \alpha\log k$ *if* $n > \alpha(k-\frac{4}{3}(\alpha+1))$;
$c_k(G) \leq \alpha(2+\log\frac{k}{3})+1$ *if* $3\alpha \leq n \leq \alpha(k-\frac{4}{3}(\alpha+1))$
and $c_k(G) \leq 2\alpha$ *if* $n \leq 3\alpha$.

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