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STOCHASTIC KNAPSACK PROBLEMS

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Abstract In this paper we study and solve two different variants of static knapsack problems with random weights: The stochastic knapsack problem with simple recourse as well the stochastic knapsack problem with probabilistic constraint. Special regard is given the corresponding continuous problems and three different problem solving methods are presented. The resolution of the continuous problems serves to provide upper bounds in a branch-and-bound framework in order to solve the original problems. Numerical results on a dataset from the literature as well as a set of randomly generated instances are given.

Keywords Static stochastic knapsack problems with random weights · simple recourse · chance constrained · expectation constrained · branch-and-bound algorithm · stochastic gradient algorithm · Arrow-Hurwicz · SOCP

1 Introduction

The *knapsack problem* has been widely studied for the last decades (Kellerer et al. (2004), Harvey M. Salkin (2006)). The problem consists in choosing a subset of items that maximizes an objective function w.r.t. a given capacity constraint. More precisely, we assume each item to have a benefit or benefit per weight unit as well as a specific weight or resource. Then, our aim is to choose a subset of items in order to maximize the total benefit w.r.t. a given capacity. There is a wide range of real life applications of the knapsack problem, amongst all transportation, finance, e.g. the purchase of commodities or stocks with a limited budged or schedule planning, where different tasks with different priority or benefit should be fulfilled in a limited time.

The knapsack problem is a combinatorial problem: each item is modeled by a binary decision variable $x \in \{0, 1\}$ with $x = 1$ if the item is chosen and 0 otherwise.

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The knapsack problem is generally linear, i.e. both the objective function and the constraints are linear. Nevertheless, it is known to be NP-hard (see (Kellerer et al., 2004)).

In the deterministic case, all parameters (item weights, benefits and capacity) are known. However, in real life problems it is not uncommon that not all of the values are determined a priori. These values can be modeled by (continuously or discretely distributed) random variables which turns the underlying problem in a *stochastic optimization problem* (for a survey on optimization under uncertainty see (Sahinidis, 2004). As the deterministic problem, the stochastic knapsack problem is at least NP-hard (see (Kellerer et al., 2004)).

In this paper, item weights are supposed to be normally distributed random variables with known mean and variance whilst capacity and benefits remain deterministic. Readers interested in the case of random returns are referred to Henig (1990), Carraway et al. (1993) as well as to Morton and Wood (1997). In the latter, the authors solve this variant of a stochastic knapsack problem using dynamic and integer programming as well as a Monte Carlo solution procedure. In all three publications, the authors solve so called *stochastic target achievement problems*. This means that, instead of maximizing the expected reward, the objective of the problem is to maximize the probability to attain a certain target.

We consider two models of stochastic knapsack problems with random weights. The first is an unconstrained problem, namely the *Stochastic Knapsack Problem with simple recourse*, while the second is a constrained stochastic knapsack problem. The problems are introduced in section 2.

There are very few publications dealing with the (exact or approximated) solution of one of the problem types addressed in this paper. A handful of publications dealing with the Stochastic Knapsack Problem with simple recourse are available such as Ağralıand Geunes (2008), (Claro and de Sousa, 2008), (Cohn and Barnhart, 1998) or (Kleywegt et al., 2001). In (Cohn and Barnhart, 1998) and (Kleywegt et al., 2001) the authors also assume the weights to be normal distributed. In the latter, a Sample Average Approximation (SAA) Method is used to solve the problem approximatively. Our work is mainly inspired by the work of (Cohn and Barnhart, 1998) who used a branch-and-bound algorithm to solve the problem exactly. In Ağralıand Geunes (2008) the authors assume the item weights to follow a Poisson distribution. Like proceeded in this paper, they solve the continuous relaxation of their problem in order to compute upper bounds for a branch-and-bound algorithm. The authors of (Claro and de Sousa, 2008) solve the problem in a very different manner. As the problem can be seen as a multi-objective optimization problem, they solve, inter alia, Conditional Value at Risk (CVaR) reformulations using a SAA method as well as tabu search related techniques.

General constrained stochastic optimization problems have been widely studied by A. Prékopa. Most of his results can be found in his book (Prekopa, 1995). However, all (or at least most) of his work has been concentrated on continuous problems. In (Goel and Indyk, 1999) the authors present approximation algorithms for three different combinatorial stochastic problems. One of these problems is a constrained stochastic knapsack problem with Poisson or exponentially distributed weights and the existence of a a polynomial approximation scheme for this problem is proved.

In this paper, we give special regard to the solution of the relaxed, i.e. continuous versions of the treated problems. Most of section 3 is dedicated to the presentation of three methods to solve these relaxations.

Two of these methods are *stochastic gradient type algorithms*. First papers on this iterative stochastic approximation methods where released in the middle of the last century (Robbins and Monro (1951), Kieper and Wolfowitz (1952)). Since then, an extensive amount of theoretical results on the convergence of the stochastic gradient algorithm and its variants has been published (Polyak (1990), L'Écuyer and Yin (1998)). The method has found many applications, particularly in machine learning and control theory. For a survey, see the books by Nevel'son and Has'minskii (1976) and by Kushner and Yin (2003).

The third problem solving method presented is based on the reformulation of the stochastic problem as an equivalent deterministic problem, more precisely as a *Second Order Cone Programming* problem (see (Boyd et al., 1998)). This special type of convex optimization problem is most efficiently solved using interior point methods (Boyd and Vandenberghe, 2004).

The results obtained by studying the relaxed problem are afterwards used to provide upper bounds in a *branch-and-bound algorithm* (see section 3.2). The branch-and-bound algorithm is one of the most common ways to solve deterministic knapsack problems. One of the first papers in which the author solved the knapsack problem using a branch and bound algorithm was (Kolesar, 1967). In (Martello and Toth, 1977) the authors present a method to calculate upper bounds for the $0-1$ knapsack problem and use them in their branch-and-bound algorithm. Recent work has been published in (Sun et al., 2007) where the authors present a branch-and-bound algorithm for the more general polynomial knapsack problem. An example for a publication that uses the branch-and-bound algorithm to solve a stochastic version of the knapsack problem is (Carraway et al., 1993).

The problems studied in this work are all *static*, i.e. the decision which items to choose is made before the stochastic parameters come to be known. Most papers on the stochastic knapsack problem study the dynamic or "on-line" variant of the problem. In the case of the *dynamic stochastic knapsack problem*, the items (e.g. their reward and/or measures) are supposed to come to be known during the decision process either directly before or after an item has been chosen. Further decisions are therefor based on the weight parameters already revealed and the decision previously made. The problem consists therefor mostly in creating an optimal decision policy. For further reading see (Lin et al., 2008), (Babaioff et al., 2007), (Kleywegt and Papastavrou, 2001), (Marchetti-Spaccamela and Vercellis, 1995) or (Ross and Tsang, 1989).

Another important field of research concerning the stochastic knapsack problem is the search for *approximation algorithms* such as proposed by (Goel and Indyk, 1999) or Klopfenstein and Nace (2006). In the latter, the authors use robust and dynamic programming to find feasible solutions for the chance constrained knapsack problem with random weights. In Dean et al. (2004) the focus lies on the comparison of adaptive and non-adaptive policies for a stochastic knapsack problem in which the size of each item is random but is revealed in the moment the item is chosen. In a recent paper by Lu (2008), the author develops an approximation scheme in a quite different way using differential equations and fluid and diffusion approximation approaches.

2 Mathematical formulations

We consider a stochastic knapsack problem of the following form: Given a set of n items. Each item has a weight that is not known in advance, i.e. the decision of which items

to choose has to be made without the exact knowledge of their weights. Therefore, we handle the weights as random variables and assume that weight χ_i of item i is independently normally distributed with mean $\mu_i > 0$ and standard deviation σ_i. Furthermore, each item has a fix reward per weight unit $r_i > 0$. The choice of a reward per weight unit can be justified by the fact that the value of an item often depend on its weight which we do not know in advance. We denote by χ, μ, σ and r the corresponding n-dimensional vectors. The aim is to maximize the expected total reward $\mathbb{E}[\sum_{i=1}^{n} r_i\chi_i x_i]$. Our knapsack problem has a fix weight capacity $c > 0$ but due to the stochastic nature of the weights the objective to respect this restriction can be interpreted in different ways.

We consider two variants of stochastic knapsack problems. The second variant is studied in two equivalent formulations:

1. **The Stochastic Knapsack Problem with simple recourse** ($SRKP$)

$$\max_{x\in\{0,1\}^n} \mathbb{E}[\sum_{i=1}^{n} r_i\chi_i x_i] - d \cdot \mathbb{E}[[g(x,\chi) - c]^+] \tag{1}$$

2. **The Constrained Knapsack Problem** (CKP)

   a) **The Chance Constrained Knapsack Problem** ($CCKP$)

$$\max_{x\in\{0,1\}^n} \mathbb{E}[\sum_{i=1}^{n} r_i\chi_i x_i] \tag{2}$$

$$\text{s.t.} \quad \mathbb{P}\{g(x,\chi) \leq c\} \geq p \tag{3}$$

   a) **The Expectation Constrained Knapsack Problem** ($ECKP$)

$$\max_{x\in\{0,1\}^n} \mathbb{E}[\sum_{i=1}^{n} r_i\chi_i x_i] \tag{4}$$

$$\text{s.t.} \quad \mathbb{E}[\mathbb{1}_{\mathbb{R}^+}(c - g(x,\chi))] \geq p \tag{5}$$

where $\mathbb{P}\{A\}$ denotes the probability of an event A, $\mathbb{E}[\cdot]$ the expectation, $\mathbb{1}_{\mathbb{R}^+}$ denotes the indicator function of the positive real interval, $g(x,\chi) := \sum_{i=1}^{n} \chi_i x_i$, $[x]^+ := \max(0,x) = x \cdot \mathbb{1}_{\mathbb{R}^+}(x)$ $(x \in \mathbb{R})$, $d \in \mathbb{R}^+$ and $p \in (0.5, 1]$ is the prescribed probability.

We call *solution vector* every $x \in \mathbb{R}^n$ such that $x = \arg\max_{x\in X_{ad}} J(x,\chi)$ where J is the objective function of one of the above maximization problems and $X_{ad} \subseteq \mathbb{R}^n$ the admissible set. We refer to the the objective function maximum value of one of these problems as *solution value*.

Throughout, we denote by f and F the density and cumulative distribution function of the standard normal distribution, respectively.

3 Problem solving methods

This section is subdivided into two subsections: In the first subsection we present three possibilities the solve the relaxed stochastic knapsack problem, one for each formulation presented in 2. In the second subsection, we use these methods to calculate upper bounds for a branch-and-bound algorithm in order to solve the corresponding combinatorial problems.

3.1 Calculating upper bounds

3.1.1 The stochastic knapsack problem with simple recourse

In this formulation, the capacity constraint has been included in the objective function by using the penalty function $[\cdot]^+$ and a penalty factor $d > 0$. This can be interpreted as follows: in the case where our choice of items leads to a capacity excess, a penalty occurs per overweight unit.

In order to simplify references to the included functions, we define

$$\psi_1(x,\chi) := \mathbb{E}[\sum_{i=1}^{n} r_i \chi_i x_i] \text{ and } \psi_2(x,\chi) := \mathbb{E}[[g(x,\chi) - c]^+]$$

i.e. our objective function becomes $J(x,\chi) = \psi_1(x,\chi) - d \cdot \psi_2(x,\chi)$.

We define a new random variable $X := g(x,\chi)$ which is normally distributed with mean $\hat{\mu} := \sum_{i=1}^{n} \mu_i x_i$, standard deviation $\hat{\sigma} := \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2}$, density function $\varphi(x) = \frac{1}{\hat{\sigma}} f(\frac{x-\hat{\mu}}{\hat{\sigma}})$ and cumulative distribution function $\Phi(x) = F(\frac{x-\hat{\mu}}{\hat{\sigma}})$. Based on these definitions, we can rewrite our objective function J in a deterministic way using the following:

$$\begin{aligned}
\mathbb{E}[[X-c]^+] &= \int_{-\infty}^{\infty} [X-c]^+ \cdot \varphi(X)\, \mathrm{d}X = \int_{c}^{\infty} (X-c) \cdot \varphi(X)\, \mathrm{d}X \\
&= \int_{c}^{\infty} X \cdot \varphi(X)\, \mathrm{d}X - c \int_{c}^{\infty} \varphi(X)\, \mathrm{d}X \\
&= \hat{\mu} \int_{c}^{\infty} \varphi(X)\, \mathrm{d}X + \hat{\sigma}^2 \int_{c}^{\infty} \varphi'(X)\, \mathrm{d}X - c \int_{c}^{\infty} \varphi(X)\, \mathrm{d}X \\
&= \hat{\sigma}^2 [\varphi(X)]_c^{\infty} + (\hat{\mu} - c)\, [\Phi(X)]_c^{\infty} = \hat{\sigma}^2 \varphi(c) + (\hat{\mu} - c)\, [1 - \Phi(c)] \\
&= \hat{\sigma} \cdot f(\frac{c-\hat{\mu}}{\hat{\sigma}}) + (\hat{\mu} - c) \cdot \left[1 - F(\frac{c-\hat{\mu}}{\hat{\sigma}})\right]
\end{aligned}$$

This leads to the deterministic equivalent objective function

$$J_{det}(x) = \sum_i r_i \mu_i x_i - d \cdot \left[\hat{\sigma} \cdot f\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right) - (c - \hat{\mu}) \cdot \left[1 - F\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right)\right]\right] \tag{6}$$

As x_i is defined on the interval $[0,1]$, $SRKP$ becomes a concave optimization problem. Due to this concavity and as the objective function handles the capacity constraint, we can apply a stochastic gradient algorithm (see Algorithm 3.1).

A stochastic gradient algorithm is an algorithm that combines both Monte-Carlo method and the gradient method often used in optimization theory. Here, the former is used to approximate the gradient of the expectation function.

Stochastic Gradient Algorithm

- Choose x^0 in $X_{ad} = [0,1]^n$
- At step $k+1$, draw $\chi = (\chi_1, ..., \chi_n)$ according to its normal distribution
- Update x^k as follows:
$$x^{k+1} = x^k + \epsilon^k r^k$$
where $r^k = \nabla j(x^k, \chi)$ and $(\epsilon^k)_{k\in\mathbb{N}}$ is a σ-sequence
- For all $i = 1, ..., n$: If $x_i^{k+1} > 1$ set $x_i^{k+1} = 1$ and if $x_i^{k+1} < 0$ set $x_i^{k+1} = 0$

Algorithm 3.1

In the case of $SRKP$, we have $j(x,\chi) = \sum_i r_i\chi_i x_i - d\cdot[g(x,\chi) - c]^+$. Clearly, j is not differentiable. We now present two of the three methods presented by Andrieu (2004) to approximate its gradient.

Integration by Parts We rewrite J using the indicator function $\mathbb{1}_{\mathbb{R}^+}$:

$$\sum_j \mathbb{E}[r_j\chi_j x_j] - d\cdot\mathbb{E}[[\sum_j \chi_j x_j - c]^+] = \sum_j \mathbb{E}[r_j\chi_j x_j] - d\cdot\mathbb{E}[\mathbb{1}_{\mathbb{R}^+}[\sum_j \chi_j x_j - c] * (\sum_j \chi_j x_j - c)] \quad (7)$$

The first method to calculate the gradient of an expectation containing an indicator function presented by Andrieu (2004) is based on integration by parts. Andrieu refers to a result concerning the indicator function $\mathbb{1}_X$, which in our case of a linear function g becomes:

Theorem 1 *Let J be defined as $J(x) := \mathbb{E}[\mathbb{1}_{\mathbb{R}^+}(g(x,\chi))]$ where $\chi \in \mathbb{R}^n$ is a random vector with density φ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^1-function. We suppose that, $\forall\chi$, $\varphi(\chi) \neq 0$. Let $\mathbb{Y}_{\mathbb{R}^+}(\cdot)$ be the primitive of $\mathbb{1}_{\mathbb{R}^+}(\cdot)$. Then, using integration by parts, we get*

$$J(x) = \mathbb{E}[\mathbb{Y}_{\mathbb{R}^+}(g(x,\chi))M_i(x,\chi)]$$

where

$$M_i(x,\chi) = \frac{1}{g'_{\chi_i}(x,\chi)}\frac{\partial \ln(g'_{\chi_i}(x,\chi)/\varphi(\chi))}{\partial\chi_i}$$

It follows

$$J'(x) = \mathbb{E}[\mathbb{1}_{\mathbb{R}^+}(g(x,\chi))g'_x(x,\chi)M_i(x,\chi) + \mathbb{Y}_{\mathbb{R}^+}(g(x,\chi))M'_{ix}(x,\chi)] \quad (8)$$

IF this theorem is the same for the function $[\cdot]^+$ (i.e. with $\mathbb{Y}_{\mathbb{R}^+}$ being a primitive of $[\cdot]^+$):

For our function $J = \psi_1 + d\cdot\psi_2$, this gives us the following expression of the gradients of ψ_1 and ψ_2:

$$\psi_1'(x,\chi) = \begin{pmatrix} \mathbb{E}[r_1\chi_1] \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{E}[r_n\chi_n] \end{pmatrix}$$

$$\psi_2'(x,\chi) = \mathbb{E}[[\sum_j \chi_j x_j - c]^+ \cdot M_i(x,\chi) \cdot \begin{pmatrix} \chi_1 \\ \cdot \\ \cdot \\ \cdot \\ \chi_n \end{pmatrix} + \mathbb{Y}_{\mathbb{R}^+}(\sum_j \chi_j x_j - c) M_i{}'_x(x,\chi)]$$

As

$$\frac{\partial\varphi(\chi)}{\partial\chi_i} = \prod_{j\neq i}\varphi(\chi_j)\cdot\frac{\partial\varphi(\chi_i)}{\partial\chi_i} = \prod_{j\neq i}\varphi(\chi_j)\cdot\left(-\frac{(\chi_i-\mu_i)}{\sigma_i^2}\varphi(\chi_i)\right) = -\frac{(\chi_i-\mu_i)}{\sigma_i^2}\varphi(\chi)$$

we have

$$\begin{aligned} M_i(x,\chi) &= \frac{1}{g'_{\chi_i}(x,\chi)}\frac{\partial\ln(g'_{\chi_i}(x,\chi)/\varphi(\chi))}{\partial\chi_i} \\ &= \frac{1}{g'_{\chi_i}(x,\chi)}\frac{\varphi(\chi)}{g'_{\chi_i}(x,\chi)}\frac{g''_{\chi_i}(x,\chi)\varphi(\chi) - g'_{\chi_i}(x,\chi)\varphi'_{\chi_i}(\chi)}{\varphi^2(\chi)} \\ &= \frac{\varphi'_{\chi_i}(\chi)}{g'_{\chi_i}(x,\chi)\cdot\varphi(\chi)} = \frac{(\chi_i-\mu_i)}{\sigma_i^2}\frac{1}{x_i} \end{aligned}$$

$$M_i{}'_x(x,\chi) = \frac{\varphi'_{\chi_i}(\chi)}{\varphi(\chi)}\frac{g''_{\chi_i x}(x,\chi)}{(g'_{\chi_i}(x,\chi))^2} = -\frac{(\chi_i-\mu_i)}{\sigma_i^2}\frac{1}{x_i^2}\cdot\nu^i$$

where we define $\nu^i \in \mathbb{R}^n$ such that ${\nu^i}_i = 1$ and ${\nu^i}_j = 0$ if $j \neq i$. So we have

$$J'(x) = \begin{pmatrix} \mathbb{E}[r_1\chi_1] \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{E}[r_n\chi_n] \end{pmatrix} - d\cdot\mathbb{E}\left[\frac{(\chi_i-\mu_i)}{x_i\cdot\sigma_i^2}([\sum_j\chi_j x_j - c]^+\chi - \frac{\mathbb{Y}_{\mathbb{R}^+}(\sum_j\chi_j x_j - c)}{x_i}\nu^i)\right]$$

As in our case $\mathbb{Y}_{\mathbb{R}^+}$ can be chosen as

$$\mathbb{Y}_{\mathbb{R}^+}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \dfrac{x^2}{2} & \text{else} \end{cases}$$

we have

$$\mathbb{Y}_{\mathbb{R}^+}(\sum_j\chi_j x_j - c) = \begin{cases} 0 & \text{if } \sum_j\chi_j x_j \leq c \\ \dfrac{(\sum_j\chi_j x_j - c)^2}{2} & \text{else} \end{cases} \tag{9}$$

Approximation by convolution (see also (Andrieu et al., 2007)) The basic idea of this method, which we call "approximation by convolution (method)", is to approximate the indicator function $\mathbb{1}_{\mathbb{R}^+}$ by its convolution with a function $h_t(x) := \frac{1}{t}h\left(\frac{x}{t}\right)$ that approximates the function of Dirac when the parameter t goes to zero. The convolution of two functions is defined as follows:

$$(\rho * h)(x) := \int_{-\infty}^{\infty} \rho(y)h(x-y)\,\mathrm{d}y$$

Using a pair, continuous and non-negative function h with $\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = 1$ having his maximum in 0, we get the following approximation of a locally integrable real valued function ρ:

$$\rho_t(x) := (\rho * h_t)(x) = \frac{1}{t}\int_{-\infty}^{\infty} \rho(y)h\Big(\frac{y-x}{t}\Big)\,\mathrm{d}y$$

In the case of $\rho = \mathbb{1}_{\mathbb{R}^+}$, we have:

$$\rho_t(x) = \frac{1}{t}\int_{0}^{\infty} h\Big(\frac{y-x}{t}\Big)\,\mathrm{d}y = \frac{1}{t}\int_{0}^{\infty} h\Big(\frac{x-y}{t}\Big)\,\mathrm{d}y$$

and so

$$(\rho_t)'(x) = \frac{1}{t^2}\int_{0}^{\infty} h'\Big(\frac{x-y}{t}\Big)\,\mathrm{d}y = -\frac{1}{t}h\Big(\frac{x}{t}\Big)$$

Based on this, we get an approximation $\nabla(j_t)_x$ of the gradient of the function j which is

$$\nabla(j_t)_x(x,\chi) = \nabla(\psi_1)_x(x,\chi) - d\cdot\left(-\frac{1}{t}\cdot h\left(\frac{g(x,\chi)}{t}\right)\cdot\chi\cdot g(x,\chi) + \mathbb{1}_{\mathbb{R}^+}(g(x,\chi))\cdot\chi\right)$$

For h, various functions may be chosen. In (Andrieu et al., 2007) the authors present some possible choices for h. For each of these functions, they compute a reference value for the mean square error of the obtained approximated gradient. It turns out that, among the presented functions, $h := \frac{3}{4}(1-x^2)\mathbb{1}_1(x)$ (where $\mathbb{1}_1$ is the indicator function for the interval $]-1,1[$) is the best choice concerning this value. This leads us to the following estimation of the gradient of j:

$$\nabla(j_t)_x(x,\chi) = r\chi + d\cdot\left(\frac{3}{4t}\left(1-\left(\frac{g(x,\chi)}{t}\right)^2\right)\mathbb{1}_1\left(\frac{g(x,\chi)}{t}\right)\cdot\chi\cdot g(x,\chi) - \mathbb{1}_{\mathbb{R}^+}(g(x,\chi))\cdot\chi\right)$$

3.1.2 The constrained knapsack problem

As presented in section 2, we consider two constrained knapsack problems, one with a chance and one an expectation constraint. As

$$\mathbb{P}\{g(x,\chi) \leq c\} = \mathbb{E}[\mathbb{1}_{\mathbb{R}^+}(c - g(x,\chi))]$$

these two considered variants of the stochastic knapsack problem are in fact equivalent.

The chance constrained knapsack problem

Generally, the chance constraint (3) does not define a convex set which makes the resolution even of continuous chance constrained problems difficult.

It has been shown by Prekopa (1995) that the set defined by constraint (3) is convex if χ has a log-concave density and g is quasi-convex. The first property can easily be proved for normal distributions and as our function g is linear, it is also (quasi-)convex. This means that the chance constraint (3) defines a convex set in the special case of a chance constrained knapsack problem with normally distributed weights.

We solve the continuous $CCKP$ by reformulating it as an equivalent, deterministic Second-order-cone-programming ($SOCP$) problem (Boyd et al., 1998). An $SOCP$ problem is an optimization problem of the following form:

$$\max_{x \in X_{ad}} \quad v^T x \tag{10}$$

$$\text{s.t. } \|Ax + b\| \leq c^T x + d \tag{11}$$

where $A \in \mathbb{R}^n \times \mathbb{R}^n$, $x, v, b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. In the following, we call a constraint of the form (11) an *SOCP-constraint*.

Let Σ be the matrix of covariances of the probability vector χ. As we assume $p > 0.5$, we get the following equivalence (see e.g. (Boyd et al., 1998))

$$\mathbb{P}\{\sum_i \chi_i x_i \leq c)\} \geq p \Longleftrightarrow \sum_i \chi_i x_i + F^{-1}(p)\|\Sigma^{1/2} x\| \leq c$$

Notice that Σ is a diagonal matrix as the weights are independently distributed. Therefore, its square root $\Sigma^{\frac{1}{2}}$ is also diagonal having the standard deviations of the random variables χ_i as diagonal nonzero components.

Based on this, the relaxed chance constrained knapsack problem becomes

$$\begin{array}{ll} \max\limits_{x \in [0,1]^n} & \mathbb{E}[\sum\limits_i r_i \chi_i x_i] \\ \text{s.t.} & \sum\limits_i \mu_i x_i + \delta \|\Sigma^{1/2} x\| \leq c \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \max\limits_{x \in [0,1]^n} & \mathbb{E}[\sum\limits_i r_i \chi_i x_j] \\ \text{s.t.} & \|\Sigma^{1/2} x\| \leq -\frac{1}{\delta} \sum\limits_i \mu_i x_i + \frac{c}{\delta} \end{array}$$

where $\delta := F^{-1}(p) > 0$.

The constraint $0 \leq x_i \leq 1$ $(i = 1, ..., n)$ of the corresponding relaxed problem can also be rewritten as an $SOCP$ constraint:

$$0 \leq x_i \leq 1 \Leftrightarrow \|A_i x\| \leq x_i \wedge \|A_i x\| \leq 1$$

where $A_i \in \mathbb{R}^{1\times n}$, $A_i[1,k] = 0\ \forall k \neq i$ and $A_i[1,i] = 1$.
Then, the $SOCP$ problem becomes:

$$\max_{x\in\mathbb{R}^n} \quad v^T x \tag{12}$$

$$\text{s.t.} \quad \|\Sigma^{1/2}x\| \leq -\frac{1}{\delta}\cdot\mu\cdot x + \frac{c}{\delta} \tag{13}$$

$$\|A_i x\| \leq \nu^i x\ , \tag{14}$$

$$\|A_i x\| \leq 1\ , \tag{15}$$

where $v := (r_1\mu_1, \ldots, r_n\mu_n)$ and $\nu^i \in \mathbb{R}^n$ such that ${\nu^i}_k = 1$ if $k = i$ and ${\nu^i}_k = 0$ otherwise.

This problem does not have any strictly feasible solution vector, as constraint (14) is always tight. This becomes problematic if we want to solve this problem using the $SOCP$ program by Boyd, Lobo, Vandenberghe ((Boyd et al., 1995)) as this software can only solve strictly feasible problems. To get a strictly feasible solution, we perform a small perturbation on the right hand side of (14) by adding ε to $\nu^i x$ such that $0 < \varepsilon << 1$.

In order to solve (12)-(15) using the $SOCP$ program by Boyd et al., we determine its dual problem and study if the latter is strictly feasible:

$$\min_{\substack{w^1\in\mathbb{R},\\ w^2,w^3,z^1,z^2,z^3\in\mathbb{R}^n}} \quad -\frac{c}{\delta}\cdot w^1 - \sum_{i=1}^{n} w_i^3$$

$$\text{s.t.} \quad \left(\Sigma^{1/2}\right)^T z^1 + \sum_{i=1}^{n} A_i^T(z_i^2 + z_i^3) - \frac{1}{\delta}\cdot\mu\cdot w^1 + \sum_{i=1}^{n}\nu^i w_i^2 = v$$

$$\|z^1\| \leq w^1$$

$$\|z_i^k\| \leq w_i^k\ , \quad k = 2,3,\ i = 1,\ldots,n$$

To find a strictly feasible solution vector for the dual problem, we reformulate it as follows:

$$\min_{\substack{w^1\in\mathbb{R},\\ w^2,w^3,z^1,z^2,z^3\in\mathbb{R}^n}} \quad -\frac{c}{\delta}\cdot w^1 - \sum_{i=1}^{n} w_i^3$$

$$\text{s.t.} \quad \sigma_i\cdot z_i^1 + z_i^2 + z_i^3 - \frac{\mu_i}{\delta}\cdot w^1 + w_i^2 = r_i\mu_i\ ,\ i = 1,...,n$$

$$\sqrt{\sum_{i=1}^{n}(z_i^1)^2} \leq w^1$$

$$|z_i^k| \leq w_i^k\ , \quad k = 2,3,\ i = 1,\ldots,n$$

At this point we first use the fact that the random weights are independently distributed. If we choose $z^1 = z_i^k = 0$ $(k = 1,2\ ,\ i = 1,\ldots,n)$ and arbitrary $w_i^3 > 0$ $(i = 1,\ldots,n)$, it is easy to find strictly feasible w^1, w_i^2 $(i = 1,\ldots,n)$.

The expectation constrained knapsack problem

In general, the expectation constrained knapsack problem can be formulated as follows:

$$\max_{x\in\{0,1\}^n} \mathbb{E}[\sum_{i=1}^{n} r_i\chi_i x_i] \tag{16}$$

$$\text{s.t.} \quad \mathbb{E}[\Theta(x,\chi)] \leq \alpha \tag{17}$$

where $\alpha \in \mathbb{R}$ and $\Theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function such that 17 represents the capacity constraint.

If constraint 17 is convex and Θ is differentiable, $ECKP$ can be solved by a stochastic Arrow-Hurwicz algorithm (see Algorithm 3.2). The stochastic Arrow Hurwicz algorithm is a stochastic gradient algorithm for solving constrained stochastic optimization problems by using Lagrangian multipliers.

Stochastic Arrow-Hurwicz Algorithm

1. Choose $x^0 \in X^{ad}$ and $\lambda^0 \in [0,\infty)$ as well as two α-sequences $(\epsilon^k)_{k\in\mathbb{N}}$ and $(\rho^k)_{k\in\mathbb{N}}$
2. Given x^k and λ^k, we draw χ_{k+1} following its normal distribution, we calculate $r^k = \nabla j(x^k,\chi^{k+1})$, $\theta^k = \nabla\Theta(x^k,\chi^{k+1})$ and we update x^{k+1} and λ^{k+1} as follows:
$$x^{k+1} = x^k + \epsilon^k(r^k + (\theta^k)^T\lambda^k)$$
$$\lambda^{k+1} = \lambda^k + \rho^k\Theta(x^{k+1},\chi^{k+1})$$
3. For all $i = 1,...,n$: If $x_i^{k+1} > 1$ set $x_i^{k+1} = 1$ and if $x_i^{k+1} < 0$ set $x_i^{k+1} = 0$
4. For all $i = 1,...,n$: If $\lambda_i^{k+1} < 0$ set $\lambda_i^{k+1} = 0$

Algorithm 3.2

As the set defined by the expectation constraint 5 is the same as the set defined by the chance constraint (3), it is also convex. With the approximation by convolution method showed in section 3.1.1 we can approximate the gradient of the constraint function $\mathbb{E}[\mathbb{1}_{\mathbb{R}^+}(c - \sum_i \chi_i x_i)]$. This allows to solve the particular $ECKP$ 4 using the stochastic Arrow-Hurwicz algorithm.

3.2 Calculating lower bounds

To calculate lower bounds of the objective function and the solution value, we use a branch-and-bound algorithm based on an algorithm by Cohn and Barnhart (1998). In 3.2.1 we explain and justify the ranking of the items using dominance relationships. Then, we present the algorithm and its variants for solving a CKP.

3.2.1 Ranking the items

In order to define the binary tree used in the branch-and-bound algorithm, we rank our items. Therefor, we introduce dominance relationships and the item are ranked

according to the number of items they dominate and, in the case where several items dominate the same number of items, by their value of $\frac{r_i^2}{\sigma_i}$.

The dominance relationships are also used to prune subtrees during the algorithm in order to decrease the number of considered nodes and evaluated branches: whenever an item is rejected, we also reject all those items that are dominated by the rejected one.

SRKP: To introduce dominance relationships in the case of $SRKP$, we consider the variations of the (deterministic equivalent) objective function (6) J_{det}.

Clearly, the increase of one of the rewards per weight unit r_j increases the objective function if and only if $x_j > 0$.

To study the variations when changing the value of $\hat{\sigma}$, we calculate the derivative of J_{det} with respect to $\hat{\sigma}$:

$$\frac{\partial J_{det}}{\partial \hat{\sigma}}(x) = -d \cdot f\left(\frac{c - \hat{\mu}}{\hat{\sigma}}\right)$$

As f is strictly positive, this shows that whenever an item is replaced by another one having the same mean and reward per weight unit but smaller variance, the value of the objective function increases. Based on this study, Cohn and Barnhart (1998) introduced two types of dominance relationships: We say that item i dominates item k if one of the following holds:

1. $\mu_i = \mu_k$, $r_i \geq r_k$, $\sigma_i \leq \sigma_k$
2. $\mu_i \leq \mu_k$, $\sigma_i \leq \sigma_k$, $r_i \cdot \mu_i \geq r_k \cdot \mu_k$

CKP: In the case of $CCKP$ and $ECKP$, it is more complicated to introduce dominance relationships as in the case of $SRKP$. This is due to the fact that modifying $\hat{\sigma}$ cannot be interpreted as easily as in the former case. The only, very special case where one can say that item i dominates item k is the following:

1. $\mu_i = \mu_k$, $\sigma_i = \sigma_k$, $r_i >= r_k$

Most of the time, the items are simply ranked by their value of r_i. This ranking gave the best results in the numerical tests (compared e.g. with the ranking used for $SRKP$ or a random ranking) but can surely be improved.

3.2.2 The branch-and-bound algorithm

Branch-and-bound algorithm 3.3 is based on the branch and bound algorithm by Cohn and Barnhart (1998). We just added step 4.

The algorithm has been constructed for $SRKP$. In order to use this algorithm to solve $CCKP$ or $ECKP$, we modify step 2 in order to respect the chance or the expectation constraint: instead of testing if the next item increases the objective function (which is the case for each item at every time), we check whether the chance or the expectation constraint is still satisfied when adding the next item. For example, in the case of $CCKP$, we calculate $\Phi(c)$ i.e. the cumulative distribution function of the probability variable $X = g(x, \chi)$. Then, depending on whether the obtained value is greater or equal than the prescribed probability, we accept or reject the item.

Branch-and-Bound Algorithm

1. Rank the items as described in section 3.2.1. This ranking defines the binary tree with the highest ranked item at the root.
2. Plunge the tree as follows: Beginning at the root of the tree, add the current item if and only if the objective function increases. Assign the maximum value of the objective function found to the variable INF. This variable stores the current lower bound of the objective function. Add the found branch to the list of branches. Set the associated upper bound SUP to infinity.
3. **If** there is no branch left on our list of branches, go to step 7.
   Else take the branch of our list of branches having the maximum objective function value. Go to step 4.
4. **If** the associated upper bound SUP is greater than the current lower bound INF, go to step 5.
   Else delete the branch from the list. Go to step 3.
5. **If** there is no accepted item left in the selected branch that does not already have a plunged or rejected subtree, delete the branch from the list. Go back to step 3.
   Else, following our ranking, choose the first accepted item that does not already have a plunged or rejected subtree. Calculate an upper bound SUP for the subtree defined by rejecting this item. Go to step 6.
6. **If** SUP $\leq$ INF, reject this subtree, go to step 5.
   Else plunge the subtree as described in 2 and add the found branch together with the value SUP to the list of branches. If the value of the objective function of this branch is greater than INF, update INF.
   Go to step 3.
7. The current value INF is the optimal solution of problem (1).

Algorithm 3.3

In step 5 the calculation of upper bounds for subtrees is realized by fixing the value of items that are higher in the tree at 1 or 0 and solving the continuous problem having the x_i of the remaining items as decision variables. In the case of $SRKP$ as well as $ECKP$ and the corresponding stochastic gradient algorithms this is easily done: at every iteration we just leave out the recalculation of the fixed x_i. In the case of the $SOCP$ reformulation of $CCKP$, we solve the $SOCP$ subproblem with respect to the index set I of the items that have not already been fixed; see subsection 3.1.2).

4 Numerical results

In this section we present our numerical results concerning the algorithms presented above. The first part contains the results of the algorithms for solving the continuous knapsack problems, namely the stochastic gradient method, the stochastic Arrow-Hurwicz algorithm as well as the algorithm by Boyd et al. to solve the $SOCP$ reformulation. The first two algorithms are implemented in C language. The third one is an open source interior point algorithm whose source code can be obtained as C- as well as $MATLAB$-code. We use the C-code. In the second part of the section, the results for the branch-and-bound algorithm are presented. It has also been implemented in C-language. All tests were carried out on an Intel PC with $1GB$ RAM.

We test our methods on the same dataset as in (Cohn and Barnhart, 1998) as well as a sample of randomly created instances for each of the chosen dimensions. The *Cohn-instance* is presented in Table 1. The last column states the value of the ratio r_i^2/σ_i used for the ranking of the items. The penalty factor used is 5. For the random

Object	**reward per weight unit** r_i	**mean of the weight** μ_i	**variance** σ_i^2	$\frac{r_i^2}{\sigma_i}$
1	2	212	47	0.583
2	2	203	21	0.873
3	3	246	42	1.389
4	2	223	21	0.873
5	2	230	15	1.033
6	1	233	10	0.316
7	2	235	11	1.206
8	2	222	33	0.696
9	1	210	36	0.167
10	2	299	42	0.617
11	2	256	25	0.800
12	3	250	19	2.065
13	1	194	24	0.204
14	3	207	22	1.919
15	1	182	14	0.267

Table 1: Values of the Cohn-instance

dataset, the weight means are generated from a normal distribution with mean 225 and standard deviation 25, the variances from a uniform distribution on the interval $[5, 50]$ and the rewards per weight unit have equal probability to be 1, 2 or 3. In the case of $SRKP$, the penalty factor is always 5 and for $CCKP$ and $ECKP$ we choose a prescribed probability of $p = 0.6$. For each dimension we created 50 instances. Table 2, 3 and 4 show the average values over these 50 instances.

The idea of creating the random instances as described come from (Cohn and Barnhart, 1998) as the authors generated their instance in the same manner. We observed that by doing so, we get very small duality gaps (see Table 4). We thereupon variated the parameters d ($d = 5, 20, 80$) and c ($c/n = 50, 133, 200$) as well as the chosen distributions (for example a uniform distribution on $[100, 350]$ or $[200, 250]$ to generate the weight means) but obtained the same small gaps of at most 2%.

In the case of $SRKP$, our stochastic gradient algorithm and the branch-and-bound algorithm involving it are compared with the method of Cohn and Barnhart. In their paper, they only present the results for one unique dataset of dimension 15. We therefor apply a variant of their method to our test instances: Cohn and Barnhart used three different heuristics to calculate upper bounds for the branch-and-bound algorithm (see (Cohn and Barnhart, 1998) for details). Following a policy, they decided which heuristic to use. Unfortunately, this policy has not been explained in detail in the paper. The results presented in our paper are therefor obtained by the following variant of their method: In the case of the continuous $SRKP$ we calculate all three upper bounds and choose the smallest as reference. During the branch-and-bound algorithm, whenever the first of the three heuristics gives an upper bound which is higher than the current lower bound, we calculate an upper bound using the second method. If this upper bound once more gives a value $SUP > INF$, we also calculate the third upper bound. This method is possibly more time consuming than the method of Cohn and Barnhart, but the number of considered nodes is equal or even smaller.

4.1 The continuous stochastic knapsack problem

An example for the convergence of the stochastic gradient method involving approximation by convolution is shown in Figure 4.1. The stopping criterion is a maximum number of 2000 iterations. As shown in the figure and confirmed by numerical tests, the best result found does not change very much (less than 1%) after iteration 500. Based on this observation, we use in the following a stopping criterion for the stochastic gradient algorithm of 500 iterations.

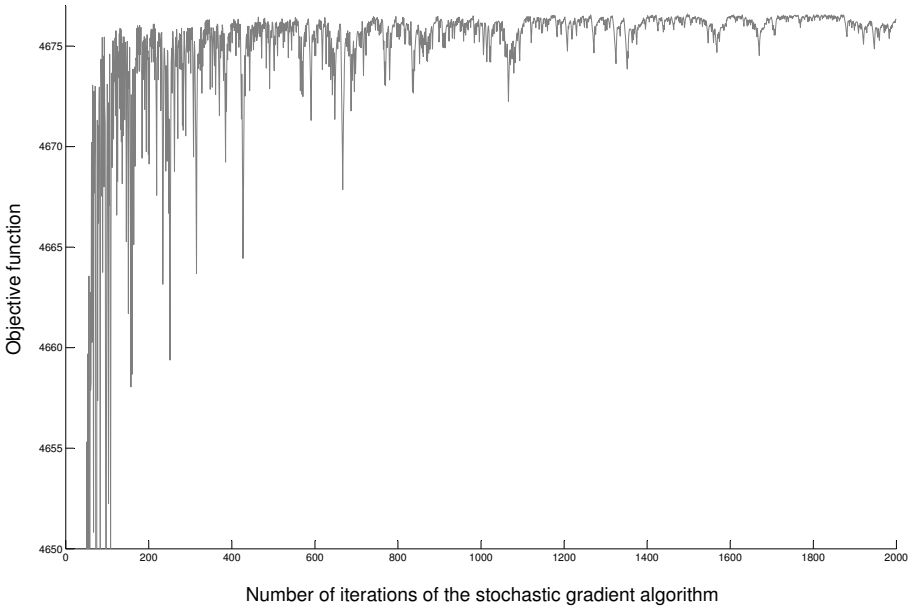

Fig. 4.1: Results for the stochastic gradient algorithm solving the continuous $SRKP$

An example for the convergence of the stochastic Arrow-Hurwicz algorithm involving approximation by convolution is presented in Figure 4.2. We solve $ECKP$ with a maximum of 8000 iterations. The first graph shows the variations of the value of the objective function whilst the second figure presents the variations of the Lagrange multiplier λ. As in the case of the stochastic gradient algorithm, we fix a maximum number of 500 iterations for all further tests.

In Table 2 and Table 3 we compare for one thing the found optima of the continuous problems, or, more precisely, the calculated upper bounds for the combinatorial problem. For another, we compare the CPU time (in milliseconds) needed to compute them. C./B. stands for Cohn/Barnhart, i.e. for the (unique) Cohn-instance of dimension 15.

Table 2 gives the results for $SRKP$. We observe that especially for small dimensions it takes much less time to compute all three upper bounds proposed by Cohn and Barnhart than to solve the continuous relaxation by a stochastic gradient algorithm.

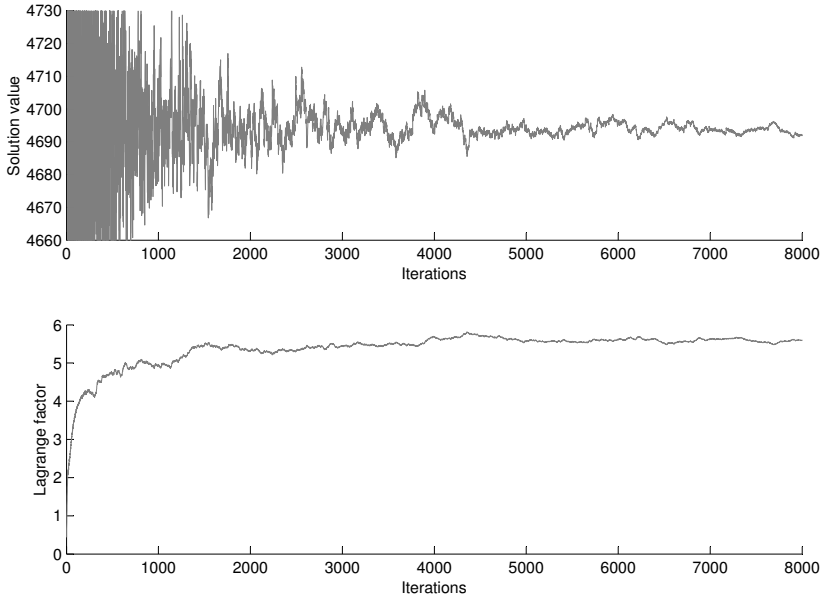

Fig. 4.2: Results for the Arrow-Hurwicz algorithm solving the continuous $ECKP$

	Stochastic gradient & Approx. by convolution		**Cohn/Barnhart**	
n	**Optimum**	**CPU-time (msec)**	**Upper Bound**	**CPU-time (msec)**
C./B.	4676.208	4	4759.000	< 1
15	4934.583	4	5146.927	< 1
20	6690.744	6	6936.017	< 1
30	10279.908	9	10529.541	< 1
50	16954.343	12	17224.803	< 1
75	25519.688	16	25811.775	< 1
100	33846.095	22	34131.754	< 1
150	50607.008	31	50932.104	< 1
250	85098.136	52	85459.649	1
500	170110.459	104	170503.708	3
1000	340922.966	240	340822.740	5
5000	1703811.095	1110	1704935.949	107
20000	6813327.586	4940	6815663.089	1759

Table 2: The numerical results for the continuous $SRKP$

But, while the CPU time of the stochastic gradient algorithm increases proportional to the dimension, this is not the case for the Cohn and Barnhart upper bounds.

Table 3 gives the results for CKP. As expected, the $SOCP$ algorithm solves the continuous CKP more accurate than the Arrow-Hurwicz algorithm, i.e. it finds a greater optimum. Concerning the CPU time, the $SOCP$ algorithm needs as much time as the Arrow-Hurwicz algorithm to solve the continuous problems of very small

	Arrow-Hurwicz & Approx. by convolution		SOCP	
n	**Optimum**	**CPU-time (msec)**	**Optimum**	**CPU-time (msec)**
C./B.	4696.097	3	4696.413	4
15	4954.546	4	4954.704	4
20	6713.081	5	6713.987	6
30	10308.640	7	10310.45	18
50	16992.450	11	16993.514	65
75	25568.059	17	25569.379	213
100	33902.283	22	33903.672	503
150	50676.686	32	50678.312	1802
250	85187.249	52	**	**
500	170239.531	107	**	**
1000	340529.019	216	**	**
5000	1704560.250	1100	**	**
20000	6814158.873	4317	**	**

** exceeding of the available memory space

Table 3: The numerical results for the continuous $CCKP/ECKP$

dimension ($n = 15, 20$). However, for higher dimensional problems the Arrow-Hurwicz algorithm is much faster than the $SOCP$ method. The $SOCP$ algorithm is also very memory space consuming: for dimensions higher than $n = 180$ the memory space of the computer used is not sufficient to solve the continuous problem using the $SOCP$ program by Boyd et al..

4.2 The combinatorial stochastic knapsack problem

The numerical results for the combinatorial problem are shown in Table 4. Notice that the CPU time needed by the branch-and-bound algorithm (columns 6 and 11) is given in seconds. Columns 5 and 10 contain the number of considered nodes, i.e. the number of times an upper bound is calculated during the branch-and-bound algorithm.

The upper table of Table 4 contains the results for $SRKP$. We observe that when using the Cohn and Barnhart upper bounds during the branch-and-bound algorithm much less nodes have to be considered. This can be explained by the higher upper bounds and therefor a smaller number of rejected subtrees. For small dimensions ($n = 15, 20, 30$) this is counterbalanced by the small CPU times needed to calculate one upper bound. In the case of higher dimensional problems, the branch-and-bound algorithm involving a stochastic gradient algorithm becomes more competitive due to the tighter upper bounds and the resulting smaller number of considered nodes.

Studying the lower table, we observe that when using the Arrow-Hurwicz algorithm a smaller number of nodes has to be considered to solve CKP than with the $SOCP$ program. This is not, as in the case of $SRKP$, due to a better choice of the upper bounds as in both algorithms the upper bounds are supposed to be the solution of the relaxed problem. Nevertheless, we get smaller values when calculating them using the Arrow-Hurwicz algorithm. This is based on the fact that the Arrow-Hurwicz algorithm involving approximation by convolution only computes approximate solutions of the

	Stochastic gradient & Approximation by convolution					Cohn/Barnhart				
n	**Upper Bound**	**CPU-time (msec) continuous**	**Optimum**	**considered nodes**	**CPU-time (sec) B-and-B**	**Upper Bound**	**CPU-time (msec) continuous**	**Optimum**	**considered nodes**	**CPU-time (sec) B-and-B**
C./B.	4676.208	4	4618	100	0.342	4759.000	< 1	4618	144	0.000
15	4934.583	4	4890	41	0.139	5146.927	< 1	4890	65	0.002
20	6690.744	6	6651	80	0.348	6936.017	< 1	6651	280	0.003
30	10279.908	9	10265	455	2.808	10529.541	< 1	10265	2525	0.037
50	16954.343	12	16951	13173	131.171	17224.803	< 1	16951	364960	779.325
75	25519.688	16	25514	63972	934.550	25811.775	< 1	*	*	*
100	33846.095	22	*	*	*	34131.754	< 1	*	*	*

* CPU-time exceeds 1h

	Arrow-Hurwicz & Approximation by convolution					SOCP				
n	**Upper Bound**	**CPU-time (msec) continuous**	**Optimum**	**considered nodes**	**CPU-time (sec) B-and-B**	**Upper Bound**	**CPU-time (msec) continuous**	**Optimum**	**considered nodes**	**CPU-time (sec) B-and-B**
C./B.	4696.097	3	4595	122	0.469	4696.413	4	4595	122	0.406
15	4954.546	4	4840	34	0.116	4954.704	4	4840	34	0.082
20	6713.081	5	6634	71	0.305	6713.987	6	6634	66	0.236
30	10308.640	7	10272	345	2.314	10310.45	18	10272	350	1.801
50	16992.450	11	16974	1880	19.473	16993.514	65	16975	7406	70.914
75	25568.059	17	25547	3743	57.397	25569.379	213	25548	62175	1535.520
100	33902.283	22	33893	94984	1932.097	33903.672	503	*	*	*
150	50676.686	32	*	*	*	50678.312	1802	*	*	*

* CPU-time exceeds 1h

Table 4: Numerical results for the combinatorial $SRKP$ (upper table) and $CCKP/ECKP$ (lower table)

relaxed problems. These non-optimal solutions have, of course, a smaller value than the optimum and the chosen "upper bounds" seem to be tighter. As the duality gaps of the chosen instances are very small, these smaller "upper bounds" have a great impact, i.e. a lot more subtrees are rejected. This can theoretically also cause the exclusion of a subtree that contains the optimal solution. Anyway, in the case of our instances, the found optima are in both cases nearly the same.

As mentioned, Table 4 only shows the results for the combinatorial problem in the case where the average needed time over all 50 instances is at most 1h. In case of the stochastic gradient algorithm involving approximation by convolution, this limit is respected when $n = 75$ but exceeded when $n = 100$. For $n = 100$, the the CPU-time is smaller or equal than $2h$ in about 78% of the cases and only 6% of the instances needed more than $24h$ to terminate. For $n = 150$, 44% of the tests finished in at most $2h$ and 56% of the instances needed not more than $24h$.

5 Conclusion

In this paper we study, solve and compare two different variants of a stochastic knapsack problem with random weights. We apply a branch-and-bound algorithm and solve continuous subproblems in order to provide upper bounds. We use a stochastic gradient method for solving the continuous stochastic knapsack problem with simple recourse ($SRKP$) and an $SOCP$ algorithm as well as a stochastic Arrow-Hurwicz algorithm for solving the constrained version of the continuous stochastic knapsack problem (CKP). In the cases of the stochastic gradient and the Arrow-Hurwicz algorithms, approximated gradients are computed using approximation by convolution.

Concerning $SRKP$, we compare the branch-and bound algorithm involving the stochastic gradient method with a method from literature (Cohn and Barnhart (1998)). The numerical tests show, that our upper bounds are much tighter, i.e. much less nodes have to be considered. This results for higher dimensional problems in a smaller CPU time as well as a smaller memory space needed. In the case of CKP, the Arrow-Hurwicz algorithm shows a better performance for higher dimensional problems as the time to compute one upper bound is smaller. In addition, the $SOCP$ algorithm requires a lot more memory space which results in an exceeding of the available memory space for high dimensional problems.

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