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# SECOND-ORDER CONE PROGRAMMING APPROACH FOR ELLIPTICALLY DISTRIBUTED JOINT PROBABILISTIC CONSTRAINTS WITH DEPENDENT ROWS

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# Second-order cone programming approach for elliptically distributed joint probabilistic constraints with dependent rows

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**Abstract**

In this paper, we investigate the problem of linear joint probabilistic constraints. We assume that the rows of the constraint matrix are dependent, the dependence is driven by a convenient Archimedean copula. Further we assume the distribution of the constraint rows to be elliptically distributed, covering normal, $t$, or Laplace distributions. Under these and some additional conditions, we prove the convexity of the investigated set of feasible solutions. We also develop an approximation scheme for this class of stochastic programming problems based on second-order cone programming.
**Keywords:** chance constrained programming; Archimedean copulas; elliptical distributions; convexity; second-order cone programming.

## 1 Introduction

We investigate the problem

$$\min c^T x \quad \text{subject to} \quad \mathbb{P}\{Tx \le h\} \ge p,\ x \in X, \tag{1}$$

where $X \subset \mathbb{R}^n$ is a deterministic closed convex set, $c \in \mathbb{R}^n$, $h = (h_1, \ldots, h_K)^T \in \mathbb{R}^K$ deterministic vectors, $T = (t_1^T, \ldots, t_K^T)^T \in \mathbb{R}^K \times \mathbb{R}^n$ a random matrix, and $p \in [0; 1]$ is a prescribed probability level. Denote

$$X(p) := \Big\{x \in X \mid \mathbb{P}\{Tx \le h\} \ge p\Big\}. \tag{2}$$

For theoretical as well as numerical purposes, it is necessary to investigate the convexity of the set $X(p)$. To do it, we will first state a convexity result for the set

$$M(p) := \Big\{x \in X \mid \mathbb{P}\left\{g_k(x) \ge \xi_k,\ k = 1, \ldots, K\right\} \ge p\Big\}, \tag{3}$$

where $\xi := (\xi_1, \ldots, \xi_K)$ is an absolutely continuous random vector and $g_k(x)$ are continuous functions. $M(p)$ is usually called as the set of feasible solutions for a continuous chance-constrained problem with random right-hand side.

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## 1.1 Survey of literature

The problem involving probability constraints was first formulated by Charnes et al. [1958] and developed by author's subsequent papers. Since the earliest papers, it was recognized that these problems of *probabilistically (or chance) constrained programming* are hard to treat, both from theoretical and computational point of view. Van de Panne and Popp [1963] proposed a solution method for a problem of type (1) with a one-row normally distributed constraint, transformed to a nonlinear constraint similar to (13). At the same time, Kataoka [1963] investigated the problem with normally distributed *individiual probabilistic constraints* with random right hand side; in the discussion he noticed that the (still individual) constraints with random matrices are also covered by his approach.

The *convexity* is widely considered as a considerable difficulty investigating chance constrained problems. Apart from simple problems presented above, the chance constrained problems often lead to a feasible solution set which is *not* convex. Various techniques and conditions were developed to encompass this issue. As an introducing citation: the model with *joint probabilistic constraints* with independent random right hand side was treated by Miller and Wagner [1965]. The convexity of their problem is assured if the probability distribution possesses a property of decreasing reversed hazard function (increasing hazard function for their maximization form of the problem). Jagannathan [1974] extended the result to the dependent case, and considered also the case of random constraint matrix with normally distributed independent rows.

The essential step was made by Prékopa [1971] introducing the notion of *logarithmically concave probability measure*. He had proven a general theorem which allowed him to introduce many convenient probability distributions (multivariate normal, Wishart, beta, Dirichlet) and derive the convexity of the feasible set for the problem with dependent random right hand side following these probability laws. The concept was further generalized by Borell [1975] and Brascamp and Lieb [1976] to $r$-concave (or $\alpha$-concave) measures and functions (namely densities and distribution functions). The generalized definition of $r$-concave function on a set, suitable also for discrete distributions, was proposed by Dentcheva et al. [2000]. We refer to Prékopa [1995], Prékopa [2003] (Chapter 5 in Ruszczyński and Shapiro [2003]), and Chapter 4 of Shapiro et al. [2009] for an exhaustive study and bibliographical references concerning convexity theory in probabilistic programming.

Despite this considerable progress, the problem of convexity remains to be a big challenge of stochastic programming, especially for the problem (1) with random matrices. The are more or less successful extensions to the grounds; for example, Prékopa et al. [2011] have recently extended the classical result Prékopa [1974], asserting that the problem is convex if the rows are independently normally distributed and the covariances matrices of the rows are constant multiples of each other. More promising direction was started by Henrion [2007] giving a complete structural description (not only the convexity) of a one-row linear chance constraint. Henrion and Strugarek [2008], introducing a notion of *$r$-decreasing density*, succeeded to relate this new notion with $r$-concavity of constraint function $g_k$ of (3) and proving that convexity of the set $M(p)$ for the case of independent random variables. The result is applied also for the problem of convexity of $X(p)$ with normally distributed independent rows, advancing so significantly the classical results. The result for right-hand side has then been extended towards dependency by Houda [2008] (see also Houda [2009]) using a variation to the strong mixing coefficient, and by Henrion and Strugarek [2011] using the theory of *copulas*. In our paper, we pursue this last direction, and prove the convexity of the set $X(p)$ for (even non-normal) distributions falling into the class of *elliptical distributions*, and using a broader class of copulas to represent the dependency of the rows in the problem (1).

Using elliptical distributions (as underlying class of probability measures), and using copulas to represent the structural dependency, is very rare in chance-constrained programming. For the first term, Henrion [2007] restricts his consideration to a very special case of one-row constraint only. Calafiore and El Ghaoui [2006] used a similar notion of $Q$-radial distribution to develop a second-order cone constraint, but again for only one-row chance constraint. Concerning the latter term, up to our knowledge and beyond the reference mentioned above, the copula theory is used only in the context of generating scenarios for multistage stochastic optimization programs. In our paper, we exploit together both notions to reformulate the problem (1) as the problem of convex optimization, and to propose an approximation scheme for this problem using the second-order cone programming method.

In our paper we start with some insights into the theory of copulas and elliptical distributions. A convexity result for right-hand sided problem (3) is given in Section 3, whereas the main convexity result and approximation schemes for (1) are formulated in Section 4.

# 2 Preliminaries

## 2.1 Dependence

To measure dependence between constraint rows, we will use the theory of copulas. In this section we mention only some basic facts about copulas. We refer to the book Nelsen [2006] for a complete introduction to the theory.

**Definition 2.1.** A *copula* is the distribution function $C : [0;1]^K \to [0;1]$ of some $K$-dimensional random vector whose marginals are uniformly distributed on [0;1].

**Proposition 2.2** (Sklar's theorem). *For any $K$-dimensional distribution function $F : \mathbb{R}^K \to [0;1]$ with marginals $F_1, \dots, F_K$, there exists a copula $C$ such that*

$$\forall z \in \mathbb{R}^K \quad F(z) = C(F_1(z_1), \dots, F_K(z_K)). \tag{4}$$

*If, moreover, $F_k$ are continuous, then $C$ is uniquely given by*

$$C(u) = F(F_1^{-1}(u_1), \dots, F_K^{-1}(u_K)). \tag{5}$$

*Otherwise, $C$ is uniquely determined on* $\operatorname{range} F_1 \times \cdots \times \operatorname{range} F_K$.

**Definition 2.3.** A copula $C$ is called *Archimedean* if there exists a continuous strictly decreasing function $\psi : [0;1] \to \mathbb{R}^+$, called *generator of* $C$, such that $\psi(1) = 0$ and

$$C(u) = \psi^{-1}\left(\sum_{i=1}^{n} \psi(u_i)\right). \tag{6}$$

If $\lim_{u\to 0} \psi(u) = +\infty$ then $C$ is called a *strict Archimedean copula* and $\psi$ is called a *strict generator*.

The inverse $\psi^{-1}$ of a generator function is continuous and strictly decreasing on $[0;\psi(0)]$ (the value of $\psi(0)$ considered as $+\infty$ if the copula is strict). Sometimes, $\psi^{-1}$ is defined as the *generalized inverse* on the whole positive half-line $[0;+\infty)$ by setting $\psi^{-1}(s) = 0$ for $s \geq \psi(0)$, losing that the strictness of the decrease property, but such a definition is not needed through the context of our paper.

**Proposition 2.4.** *Let $\psi : [0;1] \to \mathbb{R}^+$ be convex, strictly decreasing function with $\psi(1) = 0$, $\lim_{u\to 0} \psi(u) = +\infty$, and*

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \psi^{-1}(t) \geq 0 \ \forall k = 0, 1, \dots, K \ \text{and} \ \forall t \in \mathbb{R}^+.$$

*Then $\psi$ is a strict copula generator.*

**Proposition 2.5.** *Any copula generator is convex.*

Focus our attention to the set $M(p)$ defined by (3). Assume (for each $k = 1, \ldots, K$) that elements $\xi_k$ of $\xi$ have continuous distribution functions $F_k$, and the whole vector $\xi$ has joint distribution induced by the copula $C$, representing the dependence of the rows of the problem. With these assumptions, we can rewrite the set $M(p)$ as

$$M(p) = \Big\{x \in X \mid C\Big(F_1(g_1(x)), \ldots, F_K(g_K(x))\Big) \geq p\Big\}. \tag{7}$$

**Proposition 2.6.** *If the copula $C$ is Archimedean with the (strict or non-strict) generator $\psi$ then*

$$M(p) = \left\{x \in X \mid \exists y_k \geq 0 : \psi[F_k(g_k(x))] \leq \psi(p) y_k \text{ for } k = 1, \ldots, K, \ \sum_{k=1}^{K} y_k = 1\right\}. \tag{8}$$

*Proof.* It is easily seen that

$$M(p) = \left\{x \in X \mid \psi^{-1}\left(\sum_{k=1}^{K} \psi[F_k(g_k(x))]\right) \geq p\right\} = \left\{x \in X \mid \sum_{k=1}^{K} \psi[F_k(g_k(x))] \leq \psi(p)\right\} \tag{9}$$

as the generator of an Archimedean copula is strictly decreasing function, and noting that $\psi(p) \leq \psi(0)$ if the generator is not strict (the inverse $\psi^{-1}$ is strictly decreasing on $[0; \psi(0)]$). Introducing auxiliary nonnegative variables $y = (y_1, \ldots, y_K)$ with $\sum_k y_k = 1$, the inequality in (9) is equivalent to

$$\sum_{k=1}^{K} \psi[F_k(g_k(x))] \leq \psi(p) \sum_{k=1}^{K} y_k \text{ for some } y_k \geq 0 \text{ with } \sum_{k=1}^{K} y_k = 1. \tag{10}$$

Denote

$$M_I(p) = \left\{x \in X \mid \exists y_k \geq 0 : \psi[F_k(g_k(x))] \leq \psi(p) y_k \text{ for } k = 1, \ldots, K, \ \sum_{k=1}^{K} y_k = 1\right\};$$

we will show that $M(p) = M_I(p)$. Without lost of generality we assume $p < 1$ (the case $p = 1$ is obvious).

The inclusion $M_I(p) \subseteq M(p)$ is seen immediately (it is sufficient to sum up the inequalities). For the opposite direction, consider $x \in M(p)$. It is easy to show that the inequality

$$C(u) \leq \min_{k=1,\ldots,K} u_k$$

(known as Fréchet-Hoeffding upper bound) holds for any copula $C$ and any $u = (u_1, \ldots, u_K) \in [0; 1]^K$ (see e. g. Nelsen [2006]). It follows that, for $x \in M(p)$ and an Archimedean copula $C$,

$$F_k(g_k(x)) \geq \min_{j=1,\ldots,K} F_j(g_j(x)) \geq \psi^{-1}\left(\sum_{k=1}^{K} \psi[F_k(g_k(x))]\right) \geq p \ \forall k = 1, \ldots, K$$

thus,

$$\psi\Big[F_k(g_k(x))\Big] \leq \psi(p) \ \forall k = 1, \ldots, K.$$

Now, define

$$y_k := \frac{\psi\Big[F_k(g_k(x))\Big]}{\psi(p)} \text{ for } k = 1, \ldots, K-1, \qquad y_K := 1 - \sum_{k=1}^{K-1} y_k.$$

The definition of $y_k$ implies directly that $\psi\Big[F_k(g_k(x))\Big] \geq \psi(p) y_k$ for $k = 1, \ldots, K-1$; furthermore,

$$\psi\Big[F_K(g_K(x))) \geq \psi(p) - \sum_{k=1}^{K-1} \psi\Big[F_k(g_k(x))\Big] = \psi(p)\left(1 - \sum_{k=1}^{K-1} y_k\right) = \psi(p) y_K.$$

Hence, $x \in M_I(p)$. □

| Law | Characteristic generator $\varphi(t)$ | Radial density $g_s(t)$ | Normalizing constant $c_s$ |
|---|---|---|---|
| normal | $\exp\{-t/2\}$ | $\exp\left\{-\frac{1}{2}t^2\right\}$ | $(2\pi)^{-s/2}$ |
| $t$ | $*$ | $(1+\frac{1}{\nu}t^2)^{-(s+\nu)/2}$ | $(\nu\pi)^{-s/2}\frac{\Gamma((s+\nu)/2)}{\Gamma(\nu/2)}$ |
| Cauchy | $\exp\left\{-\sqrt{t}\right\}$ | $(1+t^2)^{-(s+1)/2}$ | $\pi^{-(s+1)}\Gamma(\frac{1}{2}(s+1))$ |
| Laplace | $(1+t/2)^{-1}$ | $\exp\left\{-\sqrt{2}\lvert t\rvert\right\}$ | $\pi^{-s/2}\frac{\Gamma(s/2)}{2\Gamma(s)}$ |
| logistic | $\frac{2\pi\sqrt{t}}{e^{\pi\sqrt{t}}-e^{-\pi\sqrt{t}}}$ | $\frac{e^{-t^2}}{(1+e^{-t^2})^2}$ | $*$ |

Table 1: Table of selected multivariate elliptically symmetric distributions.

## 2.2 Elliptically symmetric random vectors

The concept of elliptically (or radially) symmetric random vectors was introduced in the field of theory of probability in early seventies of 20th century to extend the class of multivariate normal distributions. A thorough survey of basic results and properties of elliptical distributions can be found in the book Fang et al. [1990].

**Definition 2.7.** An $s$-dimensional random vector $\xi$ is said to have an *elliptically symmetric distribution*, if its characteristic function is given by

$$\phi(z) := \mathbb{E}e^{iz^T\xi} = e^{iz^T\mu}\varphi(z^T\Sigma z)$$

where $\varphi$ is some scalar function (called *characteristic generator*), $\mu$ some vector (*location parameter*), and $\Sigma$ a matrix with rank $r$ (*scale matrix*). We'll write $\xi \sim Ellip_s(\mu,\Sigma;\varphi)$.

Multivariate normal distribution $N_s(\mu,\Sigma\succ 0)$ is elliptically symmetric with parameters $\left(\mu,\Sigma,\varphi(t)=e^{-\frac{1}{2}t}\right)$. Not all elliptically symmetric distributions have density, but if they have some, it must be of the form

$$f_\xi(z) = \frac{c_s}{\sqrt{\det\Sigma}}g_s\left(\sqrt{(z-\mu)^T\Sigma^{-1}(z-\mu)}\right) \tag{11}$$

where $g_s:\mathbb{R}_0^+\to\mathbb{R}^+$ (called *radial density*), $c_s>0$ is a normalization factor ensuring that $f_\xi$ integrates to one, and $\Sigma$ is required to have a full rank, i. e., to be positive definite (we denote $\Sigma\succ 0$). The radial density of the normal distribution is $g_s(t) := \exp\{-\frac{1}{2}t^2\}$ and $c_s := (2\pi)^{-s/2}$. Among many properties of elliptical distributions we note that the class of elliptical distributions is closed under affine transformations: if $\xi\sim Ellip_s(\mu,\Sigma,\varphi)$ then for any $(r\times s)$-matrix $L$ and any $r$-vector $b$, the distribution of $L\xi+b$ is $Ellip(L\mu+b, L\Sigma L^T,\varphi)$.

*Remark* 2.8. The definition of $g_s$ is unique only up to a multiplicative constant. In this view, different equivalent formulations for elliptical density appear in the literature, mostly using the notion of *density generators* $t\mapsto c_s g_s(\sqrt{t})$ instead of radial densities. Here, we have adopted the definition and the language of Paindaveine [2012].

*Remark* 2.9. In Table 2.2 we provide a selection of prominent multivariate elliptical distribution, together with their characteristic generators and radial densities. Setting the location and scale parameters to values different than $\mu=0$ and $\Sigma=I_s$, we can easily get the non-standardized versions of these well-known distributions. Note that the Cauchy distribution is a special case of $t$ distribution with $\nu=1$. The star $*$ denotes an expression which is too involved to be mentioned in the table. Concerning the multivariate $t$ distribution we refer to the book Kotz and Nadarajah [2004]; for the logistic distribution, see Volodin [1999].

The following result is a special case of Lemma 2.2 in Henrion [2007].

**Lemma 2.10.** *Assume $\xi \sim Ellip(\mu, \Sigma; \varphi)$ where $\Sigma \succ 0$, and denote*

$$Y(p) := \Big\{x \in X \mid \mathbb{P}\{\xi^T x \le h\} \ge p\Big\}. \tag{12}$$

*Then*

$$Y(p) = \Big\{x \in X \mid \mu^T x + \Psi^{-1}(p)\sqrt{x^T \Sigma x} \le h\Big\} \tag{13}$$

*where $\Psi$ is one-dimensional distribution function induced by the characteristic function $\phi(t) = \varphi(t^2)$. In particular, $\Psi$ does not depend on $x$.*

Using a variation of this lemma and Proposition 2.6, we can formulate the following result.

**Proposition 2.11.** *Suppose, in* (1)*, that $t_k^T \sim Ellip(\mu_k, \Sigma_k, \varphi_k)$ (with appropriate dimensions) where $\Sigma_k \succ 0$. Then the feasible set of the problem* (1) *can be equivalently written as*

$$X(p) = \left\{x \in X \mid \exists y_k \ge 0 : {\mu_k}^T x + \Psi_k^{-1}\left(\psi^{-1}(y_k \psi(p))\right)\sqrt{x^T \Sigma_k x} \le h_k,\ k = 1, \dots, K,\ \sum_k y_k = 1\right\} \tag{14}$$

*where $\Psi_k$ are one-dimensional distribution functions induced by the characteristic functions of the form $\phi_k(t) = \varphi_k(t^2)$, and $\psi$ is a generator of an Archimedean copula describing the dependence properties of the rows of the matrix $T$.*

*Proof.* If $x = 0$, the equivalence is obvious. Suppose so that $x \ne 0$ (say $x \notin X$) and denote

$$\xi_k(x) := \frac{t_k^T x - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}, \qquad g_k(x) := \frac{h_k - \mu_k^T x}{\sqrt{x^T \Sigma_k x}},$$

then

$$\begin{aligned} X(p) &= \Big\{x \in X \mid \mathbb{P}\Big[Tx \le h\Big] \ge p\Big\} \\ &= \Big\{x \in X \mid \mathbb{P}\Big[t_k^T x \le h_k,\ k = 1, \dots, K\Big] \ge p\Big\} \\ &= \Big\{x \in X \mid \mathbb{P}\Big[\xi_k(x) \le g_k(x),\ k = 1, \dots, K\Big] \ge p\Big\}. \end{aligned} \tag{15}$$

According to the calculus rule for elliptical distributions, namely $\phi_{c^T\xi + d}(t) = e^{itd} \cdot \phi_\xi(ct)$, the characteristic function of $\xi_k(x)$ is

$$\phi_{\xi_k(x)}(t) = \exp\left\{-it\frac{\mu^T x}{\sqrt{x^T \Sigma x}}\right\} \cdot \phi_{t_k^T}\left(\frac{x}{\sqrt{x^T \Sigma x}} t\right).$$

The characteristic function of $t_k^T$ is $\phi_{t_k^T}(z) = e^{iz^T \mu}\varphi_k(z^T \Sigma z)$, so $\phi_{\xi_k(x)}(t) = \varphi_k(t^2)$. It follows that the distribution function of $\xi_k(x)$ is $\Psi_k$, independent of $x$. Returning to (15), and applying Proposition 2.6, we have

$$\begin{aligned} X(p) &= \left\{x \in X \mid \exists y_k \ge 0 : \psi[\Psi_k(g_k(x))] \le y_k \psi(p),\ k = 1, \dots, K,\ \sum_{k=1}^K y_k = 1\right\} \\ &= \left\{x \in X \mid \exists y_k \ge 0 : g_k(x) \le \Psi_k(\psi^{-1}(\psi(p) y_k),\ k = 1, \dots, K,\ \sum_{k=1}^K y_k = 1\right\} \\ &= \left\{x \in X \mid \exists y_k \ge 0 : {\mu_k}^T x + \Psi_k^{-1}\left(\psi^{-1}(y_k \psi(p))\right)\sqrt{x^T \Sigma_k x} \le h_k,\ k = 1, \dots, K,\ \sum_k y_k = 1\right\}. \end{aligned}$$

□

The model with normally distributed rows now comes as a special case of the Proposition 2.11.

# 3 Convexity

To deal with convexity we need to describe first two special notions: that of $r$-concave function and of $r$-decreasing density.

**Definition 3.1** (Prékopa [1995], Chapter 4 of Shapiro et al. [2009])**.** A function $f\colon \mathbb{R}^s \to (0;+\infty)$ is called *$r$-concave* for some $r \in [-\infty;+\infty]$ if

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f^r(x) + (1-\lambda)f^r(y)]^{1/r} \tag{16}$$

is fulfilled for all $x, y \in \mathbb{R}^s$ and all $\lambda \in [0;1]$. The cases $r = -\infty, 0, +\infty$ are to be interpreted by continuity.

The case $r = 1$ is concavity in the usual sense. The case $r = 0$ correspond to the so-called log-concavity, i. e., to the case in which the function $\ln f$ is concave. The case $r = -\infty$ is known as quasi-concavity and corresponding right-hand side of (16) takes the form of $\min\{f(x), f(y)\}$. If $f$ is $r$-concave for some $r$, then it is $r'$-concave for all $r' \leq r$; in particular, all $r$-concave functions are quasi-concave.

**Definition 3.2** (Henrion and Strugarek [2008])**.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *$r$-decreasing* for some $r \in \mathbb{R}$ with the threshold $t^* > 0$ if it is continuous on $(0;+\infty)$ and the function $t \mapsto t^r f(t)$ is strictly decreasing for all $t > t^*$.

The threshold $t^*$ depends on the value of $r$, hence, in this view, it can be considered as a function of $r$. For simplicity, we have dropped this implicit dependence from the notation. If the function $f(t)$ is non-negative and $r$-decreasing for some $r$, then it is $r'$-decreasing for all $r' \leq r$. In particular, if $r > 0$ then $f(t)$ is 0-decreasing, hence strictly decreasing for $t > t^*$. The table of prominent one-dimensional $r$-decreasing densities together with their thresholds has been given in Henrion and Strugarek [2008]. By the following proposition, we add some elliptical to this list.

**Proposition 3.3.** *The following one-dimensional elliptical distributions have $r$-decreasing densities for some $r$:*

1. *normal distribution, for $r > 0$ with the threshold $t^* = \frac{1}{2}\left(\mu + \sqrt{\mu^2 + 4r\sigma^2}\right)$;*
2. *Student's $t$ distribution with $\nu$ degrees of freedom, for $0 < r < \nu + 1$ with the threshold $t^* = \sqrt{\frac{r\nu}{\nu+1-r}}$*
3. *Laplace (double exponential) distribution, for all $r > 0$ with the threshold $t^* = \frac{r\sigma}{\sqrt{2}}$.*

*Proof.* To test $r$-decreasing property for a differentiable elliptical density we have only to check if the derivative is strictly negative for $t > t^*$. Due to the special form (11) of elliptical density, this is equivalent to check

$$\left(\frac{\mu}{\sigma} + \hat{t}\right) g_s'(\hat{t}) + r g_s(\hat{t}) < 0, \tag{17}$$

for all $\hat{t} > \hat{t}^* \geq \mu$, using the substitution $\hat{t} := \frac{t-\mu}{\sigma}$ for $t \geq \mu$. By the backward substitution, the resulting threshold will have the form

$$t^* := \mu + \sigma \hat{t}^*. \tag{18}$$

1. The proof for normal distribution was shown in Henrion [2007] as Proposition 4.1.

2. The derivative of the radial density for (non-standardized) $t$ distribution reads as

$$g_s'(t) = -\frac{s+\nu}{\nu+t^2} t g_s(t),$$

hence the condition (17) translates to

$$-\hat{t}^2(s+\nu-r) - \hat{t}\frac{\mu}{\sigma}(s+\nu) + r\nu < 0.$$

The optimal threshold is calculated through (18) as

$$t^* = \mu\left(1 + \frac{s+\nu}{2\sigma(s+\nu-r)}\right) + \sqrt{\frac{1}{4}\left(\frac{s+\nu}{s+\nu-r}\mu\right)^2 + \frac{r\nu}{s+\nu-r}\sigma^2}$$

for $r < s+\nu$. For standardized univariate distribution use $\mu = 0, \sigma = 1, s = 1$ to get the result.

3. The condition (17) for Laplace distribution reduces to

$$-\left(\frac{\mu}{\sigma} + \hat{t}^*\right)\sqrt{2} + r < 0$$

which translates to the optimal $\hat{t}^* = \frac{r}{\sqrt{2}} - \frac{\mu}{\sigma}$; the value of $t^*$ is then easily computed by (18).

□

Concerning convexity of the set $X(p)$ with multivariate normal distributions, it is possible to exploit directly the result of Henrion and Strugarek [2011] using Gumbel copula and the proof of convexity for normally distributed random matrix from Henrion and Strugarek [2008]. But we will first prove the following modification of Theorem 1 from Henrion and Strugarek [2011], which generalizes the result for convexity of $M(p)$ in the case of Archimedean copulas (instead of author's logexp-concave copulas). It is worth to note, still, that Gaussian copulas cannot be used here as they are neither Archimedean nor generally logexp-concave.

**Theorem 3.4.** *Consider the set $M(p)$ and the following assumptions for $k = 1, \ldots, K$:*

1. *there exist $r_k > 0$ such that $g_k$ are $(-r_k)$-concave,*
2. *the marginal distribution functions $F_k$ have $(r_k+1)$-decreasing densities with the thresholds $t_k^*$, and*
3. *the copula $C$ is Archimedean with a strict generator $\psi$, and $u \mapsto \psi(e^u)$ is convex function on $(-\infty; 0]$.*

*Then $M(p)$ is convex for all $p > p^* := \max_k F_k(t_k^*)$.*

*Proof.* Let $p > p^*$, $\lambda \in [0; 1]$, and $x, y \in M(p)$. We have to show that $\lambda x + (1-\lambda)y \in M(p)$, that is

$$C\Big(F_1[g_1(\lambda x + (1-\lambda)y)], \ldots, F_K[g_K(\lambda x + (1-\lambda)y)]\Big) = \psi^{-1}\left\{\sum_{k=1}^K \psi\Big(F_k[g_k(\lambda x + (1-\lambda)y)]\Big)\right\} \geq p$$

or, equivalently,

$$\sum_{k=1}^K \psi\Big(F_k[g_k(\lambda x + (1-\lambda)y)]\Big) \leq \psi(p)$$

(c. f. the proof of Proposition 2.6). Denote, for $k = 1, \dots, K$,

$$q_k^x := F_k[g_k(x)], \qquad q_k^y := F_k[g_k(y)].$$

In the first part of the proof of Theorem 1 in Henrion and Strugarek [2011] it has been shown, based on assumptions 1 and 2, that

$$F_k[g_k(\lambda x + (1-\lambda)y)] \geq [q_k^x]^\lambda [q_k^y]^{1-\lambda},$$

hence

$$\psi\Big\{F_k[g_k(\lambda x + (1-\lambda)y)]\Big\} \leq \psi\Big\{[q_k^x]^\lambda [q_k^y]^{1-\lambda}\Big\} = \psi\Big\{\exp\Big[\lambda \ln q_k^x + (1-\lambda)\ln q_k^y\Big]\Big\}.$$

Assumption 3 allows us to continue

$$\psi\Big\{F_k[g_k(\lambda x + (1-\lambda)y)]\Big\} \leq \lambda\psi(e^{\ln q_k^x}) + (1-\lambda)\psi(e^{\ln q_k^y}) = \lambda\psi(q_k^x) + (1-\lambda)\psi(q_k^y).$$

Introducing auxiliary variables $y_k$ with $\sum y_k = 1$, and applying Proposition 2.6, we conclude on

$$\begin{aligned}\sum_{k=1}^K \psi\Big(F_k[g_k(\lambda x + (1-\lambda)y)]\Big) &\leq \sum_{k=1}^K \Big(\lambda\psi(q_k^x) + (1-\lambda)\psi(q_k^y)\Big) \\ &\leq \sum_{k=1}^K \Big(\lambda\psi(p)y_k + (1-\lambda)\psi(p)y_k\Big) = \psi(p).\end{aligned}$$

□

**Proposition 3.5.** *The following copulas satisfy assumption 3 of Theorem 3.4:*

1. *independent (product) copula with $\psi(t) = -\ln t$;*
2. *Gumbel-Hougaard copulas with $\psi_\theta(t) = (-\ln t)^\theta$ and $\theta \geq 1$;*
3. *Frank copulas with $\psi_\theta(t) = -\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ and $\theta > 0$;*
4. *Clayton copulas with $\psi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$ and $\theta > 0$.*

*Proof.* Denote $\tilde{\psi}_\theta(t) := \psi_\theta(e^t)$ for $t \in (-\infty; 0]$ and copula parameter $\theta$ (if needed).

1. The independent copula is special case of the Gumbel-Hougaard copula (see below), using $\theta = 1$.
2. For the Gumbel-Hougaard copula and $x, y \in (-\infty; 0]$,

   $$\lambda\tilde{\psi}_\theta(x) + (1-\lambda)\tilde{\psi}_\theta(y) = \lambda(-x)^\theta + (1-\lambda)(-y)^\theta \geq \Big(-\lambda x - (1-\lambda)y\Big)^\theta = \tilde{\psi}_\theta\Big(\lambda x + (1-\lambda)y\Big)$$

   as the power function is convex on $[0; +\infty)$ for $\theta \geq 1$.
3. For the Frank copulas: both the negative natural logarithm function $-\ln(\cdot)$ and the exponential function $\exp(\cdot)$ are convex functions, hence (together with their monotonicity), for $x, y \in (-\infty; 0]$,

   $$\begin{aligned}\lambda\tilde{\psi}_\theta(x) + (1-\lambda)\tilde{\psi}_\theta(y) &= -\lambda \ln\frac{\exp\{-\theta e^x\} - 1}{e^{-\theta} - 1} - (1-\lambda)\ln\frac{\exp\{-\theta e^y\} - 1}{e^{-\theta} - 1} \\ &\geq -\ln\left\{\frac{\lambda\exp\{-\theta e^x\} + (1-\lambda)\exp\{-\theta e^y\} - 1}{e^{-\theta} - 1}\right\} \\ &\geq -\ln\left\{\frac{\exp\{-\theta\lambda e^x - \theta(1-\lambda)e^y\} - 1}{e^{-\theta} - 1}\right\} \\ &\geq -\ln\left\{\frac{\exp\{-\theta e^{\lambda x + (1-\lambda)y}\} - 1}{e^{-\theta} - 1}\right\} = \tilde{\psi}\Big(\lambda x + (1-\lambda)y\Big)\end{aligned}$$

4. For the Clayton copula and $t \in (-\infty; 0]$, simply compute

$$\tilde{\psi}''(t) = \theta e^{-\theta t}$$

which is positive everywhere if $\theta > 0$.

$\square$

# 4 Main result

## 4.1 Convex reformulation

We now focus on reformulation of the feasible set $X(p)$ of the problem (1). In the following theorem we introduce sufficient conditions under which the set $X(p)$ is convex.

**Theorem 4.1.** *Consider problem* (1) *where*

1. *rows $t_k^T$ of the matrix $T$ have elliptically symmetric distributions with parameters $(\mu_k, \Sigma_k, \varphi_k)$ where $\Sigma_k$ are positive definite matrices; denote by $\Psi_k$ the (scaled row) distribution functions generated by characteristic functions of the form $\varphi_k(t^2)$;*
2. *the joint distribution function of $\Psi_k$ is driven by an Archimedean copula with a generator $\psi$.*

*Then the problem* (1) *can be equivalently written as*

$$\begin{aligned} \min c^T x \quad & \textit{subject to} \\ & {\mu_k}^T x + \Psi_k^{-1}\left(\psi^{-1}(y_k \psi(p))\right)\sqrt{x^T \Sigma_k x} \le h_k, \\ & \sum_k y_k = 1 \\ & x \in X,\ y_k \ge 0 \textit{ with } k = 1, \dots, K. \end{aligned} \tag{19}$$

*Moreover, if*

3. *the function $u \mapsto \psi(e^u)$ is convex;*
4. *the densities associated with $\Psi_k$ are (at least) 3-decreasing, with thresholds $t_k^* > 0$;*
5. *$p > p^* := \max_k \left\{\Psi_k\left(\max\{t_k^*, 4\lambda_{max}^{(k)}[\lambda_{min}^{(k)}]^{-3/2}\|\mu_k\|\}\right)\right\}$, where $\lambda_{max}^{(k)}, \lambda_{min}^{(k)}$ are largest and lowest eigenvalues of the matrices $\Sigma_k$,*

*then the problem is convex.*

*Proof.* The first part of the theorem has been already proven as Proposition 2.11. For the convexity result, we partially reproduce the proof of Theorem 5.1 of Henrion and Strugarek [2008] but with modifications concerning our use of copulas and elliptical distribution. We consider separately the cases $0 \in X(p)$ and $0 \notin X(p)$. Note that $p > 0$ by assumption 5 (as a particular case), hence the property $0 \in X(p)$ is equivalent to $h_k \ge 0$ for all $k = 1, \dots, K$.

Consider first the case $0 \notin X(p)$. Denote again

$$\xi_k(x) := \frac{t_k^T x - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}, \qquad g_k(x) := \frac{h_k - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}.$$

The (one-dimensional) random variables $\xi_k(x)$ have elliptical distributions with the distribution functions $\Psi_k$, not depending on $x$. Furthermore, our feasible set can be rewritten as

$$X(p) = \{x \in \mathbb{R}^n \mid \mathbb{P}\{\xi_k(x) \le g_k(x),\ k = 1, \dots, K\} \ge p\}.$$

Denote

$$u_k^* := 4\lambda_{max}^{(k)}[\lambda_{min}^{(k)}]^{-3/2},$$
$$\Omega^{(k)} := \left\{x \in \mathbb{R}^n \mid h_k - \mu_k^T x > u_k^* \sqrt{x^T \Sigma_k x}\right\}.$$

Together with assumption 5, it can be shown that

$$X(p) \subseteq \Omega^{(k)}. \tag{20}$$

To prove the inclusion, let $x \in X(p)$ be arbitrary. Then

$$\Psi_k(g_k(x)) \geq \min_{k=1,\ldots,K} \Psi_k(g_k(x)) \geq C\left(\Psi_1(g_1(x)), \ldots, \Psi_K(g_K(x))\right) \geq p > \Psi_k(u_k^*).$$

Due to assumption 4, $\Psi_k$ is strictly increasing at least if $p > \Psi_k(t_k^*)$ which is assured by assumption 5. Hence, $g_k(x) > u_k^*$ and thus $x \in \Omega^{(k)}$.

In Henrion and Strugarek [2008] (through Lemma 5.1 and the proof of Theorem 5.1), it was shown that the functions $g_k(x)$ are $(-2)$-concave on the set $\Omega^{(k)}$. We will not repeat the whole (a little subtle) proof here – it does rely neither on properties of copulas nor on particular characteristics of distributions, hence, it applies here without any modification. At the same time, the proof deals also with the issue that $g_k$ must not be defined and be $(-2)$-concave on the whole space (as supposed by Theorem 3.4). With identical arguments, using $(-2)$-concavity on $\Omega^{(k)}$ instead, together with assumptions 2, 4, and relation (20), is enough to apply the statement of Theorem 3.4 to this modified setting to conclude on the convexity of $X(p)$.

Consider now the case $0 \in X(p)$, i. e., all $h_k \geq 0$. Suppose $x, y \in X(p)$ arbitrary, we have to check $x_\lambda := \lambda x + (1-\lambda)y \in X(p)$ for all $\lambda \in [0; 1]$. Again, this part of the proof does not differ from the proof of Theorem 5.1 in Henrion and Strugarek [2008]:

1. If $x = y = 0$ leads to $x_\lambda = 0 \in X(p)$ by assumption.
2. If $x = 0$, $y \neq 0$, $x_\lambda \in X(p)$ by Proposition 5.1 of Henrion and Strugarek [2008] with the remark that this proposition remains valid for arbitrary distribution (not only for the normal one).
3. If $x \neq 0$, $y = 0$, $x_\lambda \in X(p)$ by the same argument.
4. If $x \neq 0$, $y \neq 0$, either $x_\lambda = 0 \in X(p)$, or $x_\lambda \neq 0$ and we can proceed as in the first part of the proof to state the $(-2)$-concavity of the function $g_k$ leading to $x_\lambda \in X(p)$ and the desired convexity result.

□

## 4.2 SOCP approximation

The formulation (19) of the problem (1) is still not a second-order cone program due to decision variables appearing as arguments to the (nonlinear) quantile functions $\Psi_k^{-1}$. To resolve the issue, we formulate lower and upper approximations to the problem (19) using favorable properties of Archimedean generators. We first formulate an auxiliary convexity lemma which gives us the possibility to find these approximations.

**Lemma 4.2.** *If*

1. *$\Psi$ is a distribution function induced by the characteristic function $\phi(t) = \varphi(t^2)$ where $\varphi$ is characteristic generator of an elliptical distribution,*

*2. the associated density is 0-decreasing with some threshold $t^* > 0$,*

*3. $p > p^* = \Psi(t^*)$, and*

*4. $\psi$ is a generator of an Archimedean copula,*

*then the function*

$$y \mapsto \Psi^{-1}\left(\psi^{-1}(y\psi(p))\right) \tag{21}$$

*is convex on $[0;1]$.*

*Proof.* Proposition 2.5 claims that $\psi$ is strictly decreasing convex function on $[0;1]$, hence $\psi^{-1}$ is strictly decreasing convex function on $[0;\psi(0)]$ with values in $[p;1]$. The second assumption implies the concavity of $\Psi(\cdot)$ on $(t^*,+\infty)$, hence the convexity of $\Psi^{-1}(\cdot)$ on $(p^*;1]$. Together with the third assumption, and the fact that $\Psi^{-1}$ is distribution function, hence non-decreasing, the assertion of lemma is proved. □

The proposed approximation technique follows the outline appearing in Cheng and Lisser [2012] and Cheng et al. [2012]. For each variable $y_k$, consider a partition of the interval $(0;1]$ in the form $0 < y_{k1} < \ldots < y_{kJ} \leq 1$.[1]

#### 4.2.1 Lower bound: piecewise tangent approximation

**Theorem 4.3.** *The optimal value of the problem*

$$\begin{aligned}
\min c^T x \quad & \textit{subject to} \\
& {\mu_k}^T x + \sqrt{z^{kT}\Sigma_k z^k} \leq h_k, \\
& z^k \geq a_{kj} x + b_{kj} w^k, \\
& \sum_k w^k = x, \\
& x \in X,\ w^k \geq 0,\ z^k \geq 0 \textit{ with } k = 1,\ldots,K,\ j = 1,\ldots,J,
\end{aligned} \tag{22}$$

*where*

$$\begin{aligned}
a_{kj} &:= H_k(y_{kj}) - b_{kj} y_{kj}, \\
b_{kj} &:= \frac{\psi(p)}{f_k(H(y_{kj}))\psi'\left(\psi^{-1}(y_{kj}\psi(p))\right)}, \\
H_k(y) &:= \Psi_k^{-1}\left(\psi^{-1}(y\psi(p))\right),
\end{aligned}$$

*and $f_k$ is the density function associated with the distribution function $\Psi_k$, is a lower bound for the optimal value of the problem (1).*

*Proof.* Fix a row $k$. The first order Taylor approximations of $H_k(y)$ at each point $y_{kj}$ of the partition are given by

$$T_{H_k(y_{kj})}(y) = H(y_{kj}) + H'(y_{kj})(y - y_{kj}).$$

Using the simple fact that

$$\begin{aligned}
\left[\psi^{-1}(y_{kj}\psi(p))\right]' &= \Psi_k\left(\Psi_k^{-1}\left[\psi^{-1}(y_{kj}\psi(p))\right]\right)' \\
&= \Psi_k'(H_k(y_{kj})) \cdot (\Psi_k^{-1})'\left(\psi^{-1}(y_{kj}\psi(p))\right) \cdot \left[\psi^{-1}(y_{kj}\psi(p))\right]'
\end{aligned}$$

[1] The number $J$ of partition points can differ for each row $k$ but, to simplify the notation and without loss of generality, we consider this number to be the same for each row $k$ through the paper.

we obtain explicitly the derivative $H_k'(y_{kj})$ in terms of $\Psi_k' = f_k$ and we can continue by

$$T_{H_k(y_{kj})}(y) = H_k(y_{kj}) + \frac{\psi(p)}{f_k(H_k(y_{kj}))} \cdot (\psi^{-1})'(y_{kj}\psi(p)) \cdot (y - y_{kj}).$$

The derivative $(\psi^{-1})'$ is obtained similar way; finally we have

$$T_{H_k(y_{kj})}(y) = H_k(y_{kj}) + \frac{\psi(p)}{f_k(H_k(y_{kj})) \cdot \psi'\left(\psi^{-1}(y_{kj}\psi(p))\right)} \cdot (y - y_{kj}) =: a_{kj} + b_{kj}y,$$

where $a_{kj}$ and $b_{kj}$ are given by Theorem 4.3. According to Lemma 4.2, the function $H_k(y)$ is convex on $(0;1]$ hence the piecewise-linear function $\max_j \{a_{kj} + b_{kj}y\}$ is a lower bound for $H_k(y)$.

Introducing auxiliary decision vectors $z^k$ fulfilling (22), and $w^k := y_k x$, the final problem formulation is a SOCP problem and the proof is then completed. □

*Remark* 4.4. The linear functions $a_{kj} + b_{kj}y$ are tangent to the (quantile) function $H_k$ at the partition points; hence the origin of the name *tangent approximation.* This approximation leads to an outer bound for feasible solution set $X(p)$.

### 4.2.2 Upper bound: piecewise linear approximation

**Theorem 4.5.** *The optimal value of the problem*

$$\begin{aligned} &\min c^T x \quad \textit{subject to} \\ &\qquad {\mu_k}^T x + \sqrt{z^{kT}\Sigma_k z^k} \le h_k, \\ &\qquad z^k \ge a_{kj}x + b_{kj}w^k, \\ &\qquad \sum_k w^k = x, \\ &\qquad x \in X,\ w^k \ge 0,\ z^k \ge 0 \textit{ with } k = 1,\dots,K,\ j = 1,\dots,J-1, \end{aligned} \tag{23}$$

*where*

$$\begin{aligned} a_{kj} &:= H_k(y_{kj}) - b_{kj}y_{kj}, \\ b_{kj} &:= \frac{H_k(y_{k,j+1}) - H_k(y_{kj})}{y_{k,j+1} - y_{kj}}, \\ H_k(y) &:= \Psi_k^{-1}\left(\psi^{-1}(y\psi(p))\right), \end{aligned}$$

*is an upper bound for the optimal value of the problem* (1)*.*

*Proof.* Fix a row $k$. The linear approximation of $H_k(y)$ for $y \in [y_{kj}; y_{k,j+1}]$ (for $j = 1,\dots,J-1$) is given by

$$L_{H_k,j}(y) = a_{kj} + b_{kj}y$$

where $a_{kj}$ and $b_{kj}$ are given by Theorem 4.5. According to Lemma 4.2, the function $H_k(y)$ is convex on $(0;1]$ hence the piecewise-linear function $\max_j \{a_{kj} + b_{kj}y\}$ is an upper bound for $H_k(y)$.

Introducing auxiliary decision vectors $z^k$ fulfilling (22), and $w^k := y_k x$, the final problem formulation, i. e., inner approximation (23), is a SOCP problem and the proof is then completed. □

# 5 Conclusions

In this paper we stated equivalent deterministic formulation of the linear chance-constrained problems with random matrix of dependent and elliptically distributed rows. The row dependence was modeled through Archimedean copulae with generators having special convexity property. Additional assumption of ellipticity for probability distribution extends usual normality assumption of row vectors. Under these assumptions, and under the assumption of $r$-decreasing probability density function for the row-normalized random variables, we proved the convexity of the feasible set for sufficiently high probabilities. Inner and outer SOCP approximation of the feasible set are then given, providing upper and lower bound for the optimal value of the problem.

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