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A New Partial Sample Average Approximation Method for Chance Constrained Problems

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Abstract In this paper, we present a new scheme of a sampling method to solve chance constrained programs. First of all, a modified sample average approximation, namely Partial Sample Average Approximation (PSAA) is presented. The main advantage of our approach is that the PSAA problem has only continuous variables whilst the standard sample average approximation (SAA) contains binary variables. Although our approach generates new chance constraints, we show that such constraints are easily tractable. Moreover, it is shown that PSAA has the same convergence properties as SAA. Finally, numerical experiments are conducted to compare the proposed approximation to SAA in order to show the strength of our new sample method.

Keywords Stochastic programming · Chance constraints · Sampling approximation

Mathematics Subject Classification (2000) 90C15 · 90C59

1 Introductions

In this paper, we focus on the following chance constrained problems:

$$\begin{aligned} & \min && f(x) && \text{(1a)} \\ (CCP) \quad & s.t. && p_0(x) := \mathbb{P}\{g_j(x,\xi) \geq 0,\ j = 1,...,m\} \geq 1-\eta && \text{(1b)} \\ & && x \in X, && \text{(1c)} \end{aligned}$$

where $f : R^n \to R$ is a convex objective function, $X \subset R^n$ is convex and represents a set of additional deterministic constraints, $\xi \in R^d$ is random vector with a distribution

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F and its support $\Xi \in R^d$, $g_j : R^n \times \Xi \to R$ are given constraint mapping functions, $\eta \in (0, 1]$ is a confidence parameter, and $p_0(x)$ is the feasible probability of $x \in X$. Constraint (1b) is called chance (or probabilistic) constraint. Furthermore, we only use a single probability constraint on all the rows in the constraints $g_j(x, \xi) \geq 0$ rather than requiring each row to be satisfied with high probability individually. Such a constraint is known as a *joint chance constraint*, whereas it is called a *individual chance constraint* when $m = 1$.

Since chance constrained programming was firstly introduced by Charnes, Cooper and Symonds in [Charnes et al., 1958], it has attracted significant attention of many researchers and practitioners as it is widely applied in a broad range of areas, such as finance, transportation and energy... [Andrieu et al., 2010, Cheng and Lisser, 2012, Cheng and Lisser, 2013, Cheng and Lisser, 2014, Dentcheva and Ruszczyński, 2006]. However, a little progress has been made for solving such problems at least for two reasons. First, the feasible set of CPP is generally nonconvex even if the set X is convex and the function $g_j(x, \xi)$ is concave in x. Second, $\mathbb{P}\{g_j(x, \xi) \geq 0, j = 1, ..., m\}$ with a fixed $x \in X$ is generally hard to compute, as it requires multi-dimensional integrations. For instance, Nemirovski and Shapiro [Nemirovski and Shapiro, 2006] stated that evaluating the distribution of a weighted sum of uniformly distributed independent random variables is NP-hard. For a comprehensive overview on theory and applications of chance constrained problems, we refer the reader to the books of Prékopa [Prékopa, 1995] and Shapiro et al. [Shapiro et al., 2009].

As chance constrained problems are computationally difficult, most of literature approaches use approximation schemes. Generally, most of solving methods on chance constrained problems are twofold: on one hand, convex or tractable approximations (see, e.g., [Nemirovski and Shapiro, 2006, Cheung et al., 2012]). In [Nemirovski and Shapiro, 2006], Nemirovski and Shapiro built a large deviation-type approximation, referred to as *Bernstein approximation*, while Cheung et al. [Cheung et al., 2012] established a new large deviation bounds for the stochastic linear matrix inequalities. The other approach consists in discretizing the probability distribution and solving the obtained equivalent deterministic problem. Precisely, there are two discretization approaches: one assumes that distributions of random variables are discrete (see, e.g., [Dentcheva et al., 2000, Luedtke et al., 2010]). Dentcheva et al.[Dentcheva et al., 2000] introduced the concept of a p-efficient point of a probability distribution and then derived various equivalent problem formulations, while Luedtke et al. [Luedtke et al., 2010] gave two strengthened formulations for linear programs with joint probabilistic constraints where only the right-hand side is random. The other one uses sample approximation methods to approximate original problems (see, e.g., [Luedtke and Ahmed, 2008, Pagnoncelli et al., 2009]), which replaces the probability distribution by its empirical distribution obtained from Monte Carlo samples of ξ. Among the sample approximations in the literature, there are three popular methods: *scenario approach* [Calafiore and Campi, 2005, Calafiore and Campi, 2006], *SAA approach* [Luedtke and Ahmed, 2008, Pagnoncelli et al., 2009], and *constraint removal approach* [Campi and Garatti, 2011, Pagnoncelli et al., 2012]. The scenario approach was firstly introduced by Califiore and Campi [Calafiore and Campi, 2005, Calafiore and Campi, 2006, Campi et al., 2009], where all the sampled constraint sets are satisfied, i.e., $g_j(x, \xi) \geq 0$, $j = 1, ..., m$ is satisfied for all the Monte Carlo samples of ξ. The main advantage of this approach is

that a feasible solution is provided when the number of samples N is large enough. Moreover, if $g_j(x,\xi), j = 1,...,m$ is concave in x, then the subsequent problem is a convex problem. However, when η is not small enough, e.g., 0.1, then the obtained solution is feasible to the problem but usually costly. In other words, for not small enough η, the scenario approach is too conservative. To overcome this drawback, an alternative sample approximation is the SAA approach where only a subset of sampled constraint sets is satisfied. When N is large enough, the SAA approach provides a near-optimal solution. Further, the obtained solution converges to the optimal solution of the original problem w.r.t. some conditions [Pagnoncelli et al., 2009]. Besides, there are different variants of SAA approach to solve chance constrained problems. Hong et al. [Hong et al., 2011] proposed to reformulate the chance constraint by a difference of convex (DC) functions, and then approximated the DC function by a sequence of convex problems and the limit of the sequence solutions is a Karush-Kuhn-Tuck (KKT) point of the original problem. Further as the functions in the DC formulations are hard to evaluated, they applied SAA approach to solve the convex problems. Another different SAA approach is proposed by Barrera et al. [Barrera et al., 2014] where the confidence parameter η is quite small, e.g., 10^{-6}. They pointed out that the existing sampling-based algorithms require an impractical number of samples to yield reasonable solutions and thus proposed a variance-reduced SAA approach by using importance sampling techniques. However, the SAA problem is still hard to solve as it is a mixed integer optimization problem with N binary variables. To avoid this drawback, the constraint removal approach [Campi and Garatti, 2011, Pagnoncelli et al., 2012] was proposed in order to provide better solutions than the scenario approach and with less CPU time than SAA approach. The constraint removal approach consists in discarding a subset of sampled constraints using a heuristic algorithm, e.g., greedy algorithm, to solve the original problem.

In this paper, we present a modified sample average approximation, namely Partial Sample Average Approximation (PSAA), to solve the chance constrained problems, which makes full use of some independence information between the random variables. In contrast to the traditional SAA approach, PSAA problem has only continuous variables whilst the standard sample average approximation (SAA) contains binary variables. Despite that new chance constraints are generated in PSAA, it is shown that such constraints are easily tractable in many cases. The remainder of this paper is organized as follows. In section 2 we present a detailed study of the scenario approach and the traditional SAA approach. In section 3 the PSAA approach is introduced as well as the underlaying idea. Convexity and convergence properties for the PSAA approach are given in section 4. In section 5, we present our numerical experiments on the different sample approximation approaches. Conclusions and future research discussions are given in the last section.

2 Scenario approach and SAA method

Under the assumption that we can sample from the distribution F, the *scenario approximation* (SA) [Calafiore and Campi, 2005, Calafiore and Campi, 2006] of chance

constrained problem (1a) is as follows:

$$\begin{aligned} \min \quad & f(x) && \text{(2a)}\\ s.t. \quad & g_j(x,\hat{\xi}^t) \geq 0, \quad j=1,\ldots,m,\ t=1,\ldots,N && \text{(2b)}\\ & x \in X, && \text{(2c)} \end{aligned}$$

where the scenarios $\hat{\xi}^1,\ldots,\hat{\xi}^N$ are assumed to be independent and sampled from the distribution F. For the quality of the approximation, One relevant result is given by Campi et al. [Campi et al., 2009] as follows:

Theorem 1 *Assume that for all the scenario samples* $\overrightarrow{\hat{\xi}} = \{\hat{\xi}^1,\ldots,\hat{\xi}^N\}$*, the scenario approximation (2) is either unfeasible, or, if feasible, it attains an unique optimal solution* $x^*(\overrightarrow{\hat{\xi}})$*. Given* $\beta \in (0,1]$*, if the sample size* N *satisfies the relation*

$$N \geq N^* := \lceil \frac{2}{\eta}(\log\frac{1}{\beta} + n) \rceil,$$

($\lceil\cdot\rceil$ *denotes the smallest integer which is greater than or equal to the argument) then,* $\mathbb{P}_{\overrightarrow{\hat{\xi}}}\{x^*(\overrightarrow{\hat{\xi}}) \text{ is either undefined or feasible for problem } (1a)\} \geq 1-\beta$.

When $X \in R^n$ is a convex set and g_j is concave in x for each ξ, the scenario approximation is a convex problem. One advantage of this approach is that there is no restriction on the distribution of ξ but only the assumption that samples can be obtained. However, Luedtke and Ahmed [Luedtke and Ahmed, 2008] pointed out that the approach is too conservative, as it requires that all sampled constraint sets be satisfied. An alternative approach is *Sample Average Approximation* (SAA) where the probability distribution is replaced by its empirical distribution obtained from Monte Carlo samples of ξ. The key difference between the two approximations is that SAA approach chooses part of sampled constraint sets, rather than all the sets, to be satisfied. Accordingly, the SAA of Problem (1a) is

$$\begin{aligned} \min \quad & f(x) && \text{(3a)}\\ s.t. \quad & \frac{1}{N}\sum_{t=1}^{N} \mathbb{I}(g_j(x,\hat{\xi}_t) \geq 0,\ j=1,\ldots,m) \geq 1-\eta && \text{(3b)}\\ & x \in X, && \text{(3c)} \end{aligned}$$

Where $\mathbb{I}(\cdot)$ is the indicator function which takes value one when $\cdot$ is true and zero otherwise. When $\eta = 0$, it becomes the scenario approach. By applying the "big-M" method, we have the equivalent formulation of the SAA problem as follows:

$$\begin{aligned} \min \quad & f(x) && \text{(4a)}\\ s.t. \quad & g_j(x,\xi^t) + My_t \geq 0,\ j=1,....m,\ t=1,...,N && \text{(4b)}\\ & \frac{\sum_{t=1}^{N} y_t}{N} \leq \eta && \text{(4c)}\\ & y_t \in \{0,1\},\ t=1,...,N, x \in X, && \text{(4d)} \end{aligned}$$

where M is large enough such that $g_j(x,\xi^t)+M \geq 0, j=1,...,m, t=1,...,N$. It is observed that the new formulation is a mixed integer problem because of the binary variables y_t. Thus, the SAA problem is hard to solve when the number of samples N becomes large.

3 Modified Sampling methods

Before we give the scheme of our approach, we present the key assumption of the modified sample method.

Assumption 1. *We assume that $\xi=(\xi_1,\xi_2)$, further ξ_1 and ξ_2 are independently distributed.*

Without loss of generality, the dimensions of vectors ξ_1 and ξ_2 are assumed to be d_1 and d_2 respectively. It is easy to see that the dimension of the vector ξ is $d=d_1+d_2$.

Lemma 1 *Let X and Y be independent integrable random variables and $g(x,y)$ be a real-valued function. If the expectation of $g(X,Y)$ exists, then*

$$\mathbb{E}[g(X,Y)]=\mathbb{E}_X\mathbb{E}_Y[g(X,Y)]=\mathbb{E}_Y\mathbb{E}_X[g(X,Y)] \tag{5}$$

Proof. Let $f_{X,Y}(x,y)$ be the joint probability density function of X and Y whereas $f_X(x)$ and $f_Y(y)$ are the marginal density functions of X and Y respectively. As X and Y are independent, then $f_{X,Y}(x,y)=f_X(x)f_Y(y)$. Further with $\mathbb{E}[g(X,Y)]=\iint g(x,y)f_{X,Y}(x,y)dxdy$, we have

$$\begin{aligned}\mathbb{E}[g(X,Y)] &= \iint g(x,y)f_X(x)f_Y(y)dxdy=\int\left(\int g(x,y)f_y(y)dy\right)f_X(x)dx\\ &=\int\left(\int g(x,y)f_x(x)dx\right)f_Y(y)dy=\mathbb{E}_X\mathbb{E}_Y[g(X,Y)]=\mathbb{E}_Y\mathbb{E}_X[g(X,Y)\end{aligned}$$

□

We reconsider the chance constraint (1b) with Assumption 1. First, the probability is represented as an expectation:

$$p_0(x)=\mathbb{P}\{G(x,\xi_1,\xi_2)\geq 0\}=\mathbb{E}[\mathbb{I}(G(x,\xi_1,\xi_2)\geq 0)]$$

where $G(x,\xi_1,\xi_2):=\min_{1\leq j\leq m} g_j(x,\xi_1,\xi_2)$. According to the results of Lemma 1, the chance constraint (1b) is equivalent to the following constraint:

$$p_0(x)=\mathbb{E}_{\xi_1}(\mathbb{E}_{\xi_2}[\mathbb{I}(G(x,\xi_1,\xi_2)\geq 0)])\geq 1-\eta$$

Applying the same idea of SAA method, we replace the distribution of ξ_1 with its empirical distribution based on Monte Carlo samples. More precisely,

$$\begin{aligned}\mathbb{E}_{\xi_1}(\mathbb{E}_{\xi_2}[\mathbb{I}(G(x,\xi_1,\xi_2)\geq 0)]) &\approx \frac{\sum_{t=1}^N \mathbb{E}_{\xi_2}[\mathbb{I}(G(x,\hat{\xi}_1^t,\xi_2)\geq 0)]}{N}\\ &=\frac{\sum_{t=1}^N \mathbb{P}\{G(x,\hat{\xi}_1^t,\xi_2)\geq 0\}}{N}\end{aligned}$$

where $\hat{\xi}_1^1,...,\hat{\xi}_1^N$ are independent Monte Carlo samples of the random variable ξ_1.

Definition 1 (**Partial sample average approximation**) Let $\hat{\xi}_1^1, ..., \hat{\xi}_1^N$ be independent Monte Carlo samples of the random variable ξ_1. Then, we have a sampled program of CPP as follows:

$$\min \quad f(x) \tag{9a}$$

$$s.t. \quad p_N(x) := \frac{\sum_{t=1}^{N} \mathbb{P}\{G(x, \hat{\xi}_1^t, \xi_2) \geq 0\}}{N} \geq 1 - \eta \tag{9b}$$

$$x \in X, \tag{9c}$$

which is equivalent to

$$V(PSAA) := \min \quad f(x) \tag{10a}$$

$$s.t. \quad \mathbb{P}\{g_j(x, \hat{\xi}_1^t, \xi_2) \geq 0, \, j = 1, ..., m\} \geq y_t, \; t = 1, ..., N \tag{10b}$$

$$(PSAA) \quad \frac{\sum_{t=1}^{N} y_t}{N} \geq 1 - \eta \tag{10c}$$

$$y_t \geq 0; t = 1, ..., N, x \in X, \tag{10d}$$

Compared to SAA approach[Luedtke and Ahmed, 2008, Pagnoncelli et al., 2009], the main advantage of our approach is that PSAA problem has only continuous variables whilst SAA problem contains binary variables though new chance constraints are generated in PSAA problem. We show that such constraints are easily tractable in some cases. However, both PSAA and SAA share the same idea that the original distribution of the random vector is replaced by its empirical distribution obtained from N independent samples.

PSAA problem becomes easier to solve if variable y_t is fixed. In order to seek a tradeoff between solution quality and computation hardness, we propose a new approach based on the PSAA method, where $y_t, t = 1, ..., N$ have the same value, i.e., $y_t = 1 - \eta, t = 1, ..., N$.

Definition 2 (**Partial scenario approach**) Let $\hat{\xi}_1^1, ..., \hat{\xi}_1^N$ be an independent Monte Carlo samples of the random variable ξ_1. Then we have a sampled program of CPP as follows:

$$V(PSA) := \min \quad f(x) \tag{11a}$$

$$s.t. \quad \mathbb{P}\{g_j(x, \hat{\xi}_1^t, \xi_2) \geq 0, \, j = 1, ..., m\} \geq 1 - \eta, \; t = 1, ..., N \tag{11b}$$

$$(PSA) \quad x \in X, \tag{11c}$$

which is called "Partial Scenario approach" (PSA).

With PSA method, chance constraint (1b) is divided into N chance constraints. It is easy to check that PSA problem is a conservative approximation of the PSAA problem, i.e., $V(PSA) \geq V(PSAA)$.

4 Discussion on the two approaches

In this section, we discuss in more details on Assumption 1 and the two approaches as well as the formulations of a special case.

4.1 Independence assumptions

The independence assumption which states that ξ is composed of two independent parts ξ_1 and ξ_2 is not restrictive in many cases. For instance, when ξ is normally distributed with mean vector μ and covariance matrix Σ, then $\xi = \mu + Az$ where $\Sigma = AA^T$ and $z = (z_1, \ldots, z_d)^T$ is a vector whose components are independent standard normal variables. Thus by replacing ξ by $\mu + Az$ in chance constraint (1b), the new random vector within the constraints satisfies the assumption.

4.2 Convexity results of PSAA

It is well known that the feasible set of Problem (1a) is generally nonconvex even if the set X is convex and the function $g_j(x, \xi)$ is concave in x. However, there are still some convexity results on chance constrained problems under given assumptions. For instance, under multivariate normal distribution, an individual chance constraint of bilinear model is a second-order constraint, which is convex, when $\eta \leq 0.5$. Here we present one of the most general results of convexity given by Shapiro et al. [Shapiro et al., 2009]:

Theorem 2 *Let $g_j(x, y)$, $j = 1, \ldots, m$ be quasi-concave functions on R^{n+d} where x is an n-dimensional vector and y is a d-dimensional vector. If $\xi \in R^d$ is a random vector that has a α-concave probability distribution where $\alpha \in [-\infty, \infty]$, then the function*

$$H(x) = \mathbb{P}\{g_j(x, \xi) \geq 0, \quad j = 1, \ldots, m\} \tag{12}$$

is α-concave on the set

$$D = \{x \in R^n : \exists \bar{\xi} \in R^d \text{ such that } g_j(x, \bar{\xi}) \geq 0, \quad j = 1, \ldots, m\}.$$

Accordingly, under the assumptions of Theorem 2, a corollary on the convexity follows.

Corollary 1 *Assume that $g_j(x, y) : R^n \times R^d$, $j = 1, \ldots, m$ be quasi-concave functions and $\xi \in R^d$ is a random vector that has an α-concave probability distribution. Then the following set is convex and closed*

$$\{x \in R^n : \mathbb{P}\{g_j(x, \xi) \geq 0, \quad j = 1, \ldots, m\} \geq p\} \tag{13}$$

where $p \in (0, 1]$.

According to Theorem 2, when ξ_2 has a logconcave probability distribution and $g_j(x, \hat{\xi}_1^t, \xi_2)$ is quasi-concave functions of (x, ξ_2), $\mathbb{P}\{G(x, \hat{\xi}_1^t, \xi_2) \geq 0\}$ is logconcave. However $p_N(x)$ is not logconcave, as logconcavity does not carry over from terms to their sum [Prékopa, 2001, Prékopa, 2003], whereas concavity does carry over. Therefore, to explore the convexity of PSAA problem, we seek the concavity of $\mathbb{P}\{G(x, \hat{\xi}_1^t, \xi_2) \geq 0\}$.

4.2.1 Preliminaries

Before presenting the convexity results, we introduce a new definition as follows:

Definition 3 (**Concave point**) z^* is said to be a *concave point* of a random variable ξ, if the cumulative distribution function $\Phi(z)$ of ξ is concave for all $z \geq z^*$.

Evidently, if z^* is a concave point of ξ, then z' is also a concave point whenever $z' \geq z^*$. When ξ is univariate and has the standard normal distribution, then 0 is a concave point of ξ and further it is the minimal concave point. For the other univariate distributions, the minimal concave point is listed in Table 1.

Table 1 Table of selected univariate distributions

Distribution	**Density function**	z^*
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(z-\mu)^2}{2\sigma^2})$	μ
Exponential	$\lambda\exp(-\lambda z) \quad z \geq 0$	0
Uniform	$\frac{1}{b-a} \quad a \leq z \leq b$	a
Welbull	$abz^{b-1}\exp(-az^b) \quad z \geq 0$	$0 \quad 0 < b \leq 1$ $(\frac{b-1}{ab})^{\frac{1}{b}} \quad 1 < b$
Gamma	$\frac{z^{k-1}e^{-\frac{z}{\theta}}}{\theta^k\Gamma(k)} \quad z > 0$	$0 \quad 0 < k \leq 1$ $(k-1)\theta \quad 1 < k$
Student	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	0

Although Table 1 collects many minimal concave point for many popular different distributions, it is hard to determine z^* for the multivariate variables. The following relevant result was given by Prékopa [Prékopa, 2001, Prékopa, 2003]:

Theorem 3 *$\Phi(z_1, ..., z_n; R)$ is concave in the set $\{z | z_i \geq \sqrt{n-1}, i = 1, ..., n\}$, where $\Phi(z_1, ..., z_n; R)$ is the n-variate standard normal probability distribution function with correlation matrix R.*

According to Theorem 3, $(\sqrt{n-1}, ..., \sqrt{n-1})$ is a concave point of n-variate standard normally distributed variables with any correlation matrix R. However it is difficult to find the minimal (or Pareto) concave point even for normally distributed vector.

We reconsider the chance constraint (9b) with the following form:

$$\mathbb{P}\{g_j(x, \xi_1^t) \geq \xi_{2_j} \geq 0, \quad j = 1, \ldots, m\} \geq 1 - \eta. \tag{14}$$

Theorem 4 *Assume that $g_j(x, \hat{\xi}_1^t)$ $j = 1, \ldots, m$ be concave functions of x and there exists a concave point $z^* = (z_1^*, ..., z_m^*)$ of ξ_2, we have $g_j(x, \hat{\xi}_1^t) \geq z_j^*$ for any $x \in X$. Then the following set is convex*

$$X_0(\xi_1^t) := \{(x; y_t) \in R^{n+1} : \mathbb{P}\{g_j(x, \hat{\xi}_1^t) \geq \xi_{2_j}, \quad j = 1, \ldots, m\} \geq y_t, x \in X\}. \tag{15}$$

Therefore the feasible set of problem PSAA is convex.

Proof. Let $G_0(x) := \mathbb{P}\{g_j(x, \hat{\xi}_1^t) \geq \xi_{2_j}, \quad j = 1, \ldots, m\}$ and $\Phi(z)$ be the cumulative distribution function of ξ_2 , which is non-decreasing function. Based on Definition 3, $\Phi(z)$ is concave when $z \geq z^*$. Further as $g_j(x, \hat{\xi}_1^t)$ is concave function of x and $g_j(x, \hat{\xi}_1^t) \geq z_j^*$ for any $x \in X$, thus $G_0(x)$ is concave on X. For any two points (x_1, y_{t_1}) and (x_2, y_{t_2}) in the set $X_0(\xi_1^t)$, and $\lambda \in [0, 1]$, we have

$$G_0(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda G_0(x_1) + (1 - \lambda)G_0(x_2) \geq \lambda y_{t_1} + (1 - \lambda)y_{t_2}$$

which concludes the proof. □

Example

We consider a simple example to verify Theorem 4 as follows:

$$\begin{aligned} \min \quad & c^T x && \text{(16a)} \\ s.t. \quad & \mathbb{P}\{\xi_1^T x \geq \xi_2\} \geq p && \text{(16b)} \\ & x \geq 0 && \text{(16c)} \end{aligned}$$

where $c \in R^n$ is a deterministic cost vector, $\xi_1 \in R^n$ and $\xi_2 \in R$ are random and both are assumed uniformly distributed. If the lower bound of ξ_2 is 0 and the lower bound of $\xi_1 \geq 0$, then its corresponding PSAA problem is a convex problem, as the minimal concave point of ξ_2 is 0.

4.3 Comparison between PSAA and SAA

Since PSAA and SAA problems have the same underlaying idea, i.e., the original distribution of the random vector is replaced by its empirical distribution obtained from N independent samples, They have the same basic properties, namely convergence properties. In the following, we present some results on convergence property for PSAA, whereas there are similar results about convergence property for the SAA problem in the paper of Pagnoncelli et al. [Shapiro et al., 2009, Pagnoncelli et al., 2009]. First of all, we denote by f^*, f_{SAA}^* and f_{PSAA}^* the optimal values of the original problem, the SAA and the PSAA problem respectively, while S, S_{SAA} and S_{PSAA} are the set of the optimal solutions defined accordingly. We modifiedrecall two definitions.

Definition 4 Function f_N is said to epiconverge to f, written as $f_N \xrightarrow{e} f$, if for a.e. $\omega \in \Omega$ the functions $f_N(\cdot, \omega)$ epiconverge to $f(\cdot)$.

Definition 5 Function $g(x,\xi)$ is said to be a *Carathéodory function*, i.e., $g(x,\xi)$ is measurable for every $x \in R^n$ and $g(x,\xi)$ is continuous for a.e. $\xi \in \Xi$.

Proposition 1 *Let* $G(x,\xi_1,\xi_2) := \min_{1\le j\le m} g_j(x,\xi_1,\xi_2)$ *be a* Carathéodory *function. Then* $p_N(x) \xrightarrow{e} p_0(x)$ *w.p.1. Furthermore, suppose that there is an optimal solution* x^* *of the original problem (i.e., problem (1a)), such that for any* $\epsilon > 0$*, there is* $x \in X$ *with* $\|x - x^*\| \le \epsilon$ *and* $p_0(x) > 1 - \eta$*, the set* X *is compact and the objective function* $f(x)$ *is continuous. Then* $f^*_{PSAA} \to f^*$ *and* $\mathbf{D}(S_{PSAA}, S) \to 0$ *w.p.1 as* $N \to \infty$*, where* $\mathbf{D}(A, B)$ *denotes the deviation of set* A *from set* B*.*

Proof. Let $\bar{p}_0(x) = 1 - p_0(x)$, $\bar{p}_N(x) = 1 - p_N(x)$ and, $\bar{S}$ and $\bar{S}_{PSAA}$ be the complements of sets S and S_{PSAA} respectively. We first prove that $\bar{p}_N(x)$ is lower semicontinuous. Since the indicator function on the open set is semicontinous and $G(x,\hat{\xi}_1^t,\xi_2)$ is a *Carathéodory* function, then $\mathbb{I}(G(x,\hat{\xi}_1^t,\xi_2) < 0)$ is random lower semicontinuous. Then following Fatou's lemma (see Theorem 7.51 in [Shapiro et al., 2009]), $\mathbb{E}_{\xi_2}[\mathbb{I}(G(x,\hat{\xi}_1^t,\xi_2) < 0)]$ is lower semicontinuous. By Applying Fatou's lemma again, $\bar{p}_N(x) = 1 - p_N(x) = \frac{\sum_{t=1}^N \mathbb{E}_{\xi_2}[\mathbb{I}(G(x,\hat{\xi}_1^t,\xi_2)<0)]}{N}$ is lower semicontinuous. As $0 \le \mathbb{E}_{\xi_2}[\mathbb{I}(G(x,\hat{\xi}_1^t,\xi_2) < 0)] \le 1$ and N samples are independent and identically distributed, then we have $\bar{p}_N(x) \xrightarrow{e} \bar{p}_0(x)$ w.p.1 based on the results of Theorem 7.51 in [Shapiro et al., 2009]. Therefore, $p_N(x) \xrightarrow{e} p_0(x)$ w.p.1.

For the remaining proof, it follows the same procedure of the proof of Proposition 5.30 in [Shapiro et al., 2009] (or proof of Proposition 2.2 in [Pagnoncelli et al., 2009]). Thus we refer to the reader to [Shapiro et al., 2009] or [Pagnoncelli et al., 2009] for more details. □

4.4 Special case

In this subsection, we present a special case of problem (1) where its subsequent PSAA approximation is tractable.

A bilinear chance constrained problem is considered with the following formulation:

$$\min \quad f(x) \tag{17a}$$
$$s.t. \quad \mathbb{P}\{M(x)\xi_1 \ge \xi_2 \mathbf{e}_m\} \ge 1 - \eta \tag{17b}$$
$$x \in X \tag{17c}$$

where $M(x) \in R^{m\times d_1}$ is a affine matrix of x, $\mathbf{e}_m \in R^m$ is all-ones vector, $\xi_1 \in R^{d_1}$ and $\xi_2 \in R$ are random variables.

Theorem 5 *If* ξ_2 *is uniformly distributed on the interval* $[L, U]$*, then the corresponding PSAA problem of the bilinear problem has a conservative (or safe) approximation*

as follows:

$$\min \quad f(x) \tag{18a}$$

$$s.t. \quad \frac{M(x)\hat{\xi}_1^t - L\mathbf{e}_m}{U-L} \geq y_t\mathbf{e}_m,\ t = 1, ..., N \tag{18b}$$

$$\frac{\sum_{t=1}^N y_t}{N} \geq 1-\eta \tag{18c}$$

$$y_t \leq 1; t = 1, ..., N \tag{18d}$$

$$x \in X \tag{18e}$$

Moreover, if there is an optimal solution of the PSAA problem denoted by x^{PSAA} such that for any sample $\hat{\xi}_1^t, t = 1, ..., N$ such that $M(x^{PSAA})\hat{\xi}_1^t \geq L\mathbf{e}_m$, then x^{PSAA} is also the optimal value of the conservative problem.

Proof. Let (x^{fea}, y^{fea}) be a feasible solution of problem (18), i.e, the conservative approximation problem. With the PSAA method, the chance constraint (17b) is approximated by the following constraint:

$$\frac{\sum_{t=1}^N \mathbb{P}\{M(x)\hat{\xi}_1^t \geq \xi_2\mathbf{e}_m\}}{N} \geq 1-\eta \tag{19}$$

Let $z_t = \mathbb{P}\{M(x^{fea})\hat{\xi}_1^t \geq \xi_2\mathbf{e}_m\}$. Since ξ_2 is uniformly distributed on the interval $[L, U]$, then we have:

$$z_t = \begin{cases} 0 & \text{if } \min\{M(x^{fea})\hat{\xi}_1^t\} < L \\ \frac{\min\{M(x^{fea})\hat{\xi}_1^t\}-L}{U-L} & \text{if } L \leq \min\{M(x^{fea})\hat{\xi}_1^t\} \leq U \\ 1 & \text{if } U \leq \min\{M(x^{fea})\hat{\xi}_1^t\} \end{cases} \tag{20}$$

Further, as (x^{fea}, y^{fea}) is a feasible solution of problem (18), we have $y_t^{fea} \leq 1$, $\frac{\sum_{t=1}^N y_t^{fea}}{N} \geq 1-\eta$ and $\frac{M(x^{fea})\hat{\xi}_1^t - L\mathbf{e}_m}{U-L} \geq y_t^{fea}\mathbf{e}_m$. Thus, we conclude that $z_t \geq y_t$, which leads to the conclusion that $\frac{\sum_{t=1}^N z_t}{N} \geq 1-\eta$. Therefore, x^{fea} is also a feasible solution of the PSAA problem and problem (18) is a conservative approximation of PSAA problem.

Let x^{PSAA} be an optimal solution of the PSAA problem which satisfies the constraint $M(x^{PSAA})\hat{\xi}_1^t \geq L\mathbf{e}_m$. Let $\pi_t = \mathbb{P}\{M(x^{PSAA})\hat{\xi}_1^t \geq \xi_2\mathbf{e}_m\}$. Then we have $\pi_t \leq \min\{1, \frac{\min\{M(x^{PSAA})\hat{\xi}_1^t\}-L}{U-L}\}$ and $\frac{\sum_{t=1}^N \mathbb{P}\{\pi_t\}}{N} \geq 1-\eta$. Thus $(x^{PSAA}, \pi_1, ..., \pi_N)$ is a feasible solution of problem (18). Moreover, it is shown that problem (18) is a conservative approximation of the PSAA problem. Therefore, $(x^{PSAA}, \pi_1, ..., \pi_N)$ is the optimal solution of problem (18). □

Note that there is no assumption on the distribution of ξ_1. Moreover, the corresponding SAA problem of the bilinear problem is a mixed integer linear problem.

4.5 Convexity results of PSA

The variables y_t are fixed in PSA problem, i.e., $y_t = 1 - \eta$. Therefore, PSA has more general convexity results than PSAA as shown by the following corollary:

Corollary 2 *Assume that $g_j(x, \hat{\xi}_1^t, \xi_2) : R^n \times R^{d_2}$, $j = 1, \ldots, m$ be quasi-concave functions and $\xi_2 \in R^{d_2}$ is a random vector that has an α-concave probability distribution. Then the following set is convex*

$$X_1(\hat{\xi}_1^t) := \{x \in R^n : \mathbb{P}\{g_j(x, \hat{\xi}_1^t, \xi_2) \geq 0,\ j = 1, \ldots, m\} \geq 1 - \eta\}. \tag{21}$$

Therefore the feasible set of PSA problem is convex.

4.6 Comparison between PSA and SA

In parallel with Theorem 1 for SA method, we also give a prior estimation of sample size N such that the optimal solution of the PSA problem is feasible to the original problem with high probability.

Theorem 6 *For the PSA problem with a confidence parameter $\eta' < \eta$, we assume that X is a convex and closed set, $f(x)$ is linear, i.e., $f(x) = c^T x$, $G_0(x, \xi_1) := \mathbb{P}_{\xi_2}\{g_j(x, \xi_1, \xi_2) \geq 0, j = 1, ..., m\}$ is continuous and convex in x for any $\xi \in \Xi_1$. Further, assume that for all the scenario samples $\overrightarrow{\hat{\xi}_1} = \{\hat{\xi}_1^1, \ldots, \hat{\xi}_1^N\}$, the PSA problem (11) is either unfeasible, or it has an unique optimal solution $x^*(\overrightarrow{\hat{\xi}_1})$ if feasible. Given $\beta \in (0, 1]$ and $\overline{\eta} = \frac{\eta - \eta'}{1 - \eta'}$, if the sample size N satisfies the relation*

$$N \geq N^* := \lceil \frac{2}{\overline{\eta}}(\log \frac{1}{\overline{\eta}} + n) \rceil,$$

then, $\mathbb{P}_{\overrightarrow{\hat{\xi}_1}}\{x^*(\overrightarrow{\hat{\xi}_1})$ is either undefined or feasible for problem $(1a)\} \geq 1 - \beta$.

Proof. First, we reformulate the PSA problem with a confidence parameter η' as follows:

$$\begin{aligned} \min \quad & c^T x && \text{(22a)} \\ s.t. \quad & 1 - \eta' - G_0(x, \hat{\xi}_1^t) \leq 0,\ t = 1, ..., N && \text{(22b)} \\ & x \in X && \text{(22c)} \end{aligned}$$

Next, following the idea of the scenario approach, the original problem of problem (22) is

$$\begin{aligned} \min \quad & c^T x && \text{(23a)} \\ s.t. \quad & \mathbb{P}_{\xi_1}\{1 - \eta' - G_0(x, \xi_1) \leq 0\} \geq 1 - \overline{\eta} && \text{(23b)} \\ & x \in X && \text{(23c)} \end{aligned}$$

We denote the optimal solution of problem (22) by $x^*(\overrightarrow{\hat{\xi}_1})$. According to Theorem 1,

$$N \geq N^* := \lceil \frac{2}{\bar{\eta}}(\log \frac{1}{\bar{\eta}} + n) \rceil,$$

then $\mathbb{P}_{\overrightarrow{\hat{\xi}_1}}\{x^*(\overrightarrow{\hat{\xi}_1})$ is either undefined or feasible for problem (23)$\} \geq 1-\beta$. When $x^*(\overrightarrow{\hat{\xi}_1})$ is feasible for problem (23), then one has

$$\mathbb{P}_{\xi_1}\{1-\eta' - G_0(x^*(\overrightarrow{\hat{\xi}_1}), \xi_1) \leq 0\} \geq 1-\bar{\eta}$$

By introducing auxiliary random variable $\delta = G_0(x^*(\overrightarrow{\hat{\xi}_1}), \xi_1) \geq 0$, we have

$$\mathbb{P}_{\xi_1}\{1-\eta' - G_0(x^*(\overrightarrow{\hat{\xi}_1}), \xi_1) \leq 0\} = \mathbb{P}_{\delta}\{1-\eta' \leq \delta\} \leq \frac{\mathbb{E}_{\delta}[\delta]}{1-\eta'}$$

where the last inequality is the *Markov inequality*. Thus, it follows that $\frac{\mathbb{E}_{\delta}[\delta]}{1-\eta'} \geq 1-\bar{\eta}$. Furthermore as $\mathbb{E}_{\delta}[\delta] = \mathbb{E}_{\xi_1}[G_0(x^*(\overrightarrow{\hat{\xi}_1}), \xi_1)] = \mathbb{P}\{g_j(x^*(\overrightarrow{\hat{\xi}_1}), \hat{\xi}_1, \xi_2) \geq 0, j = 1, ..., m\}$, then

$$\mathbb{P}\{g_j(x^*(\overrightarrow{\hat{\xi}_1}), \hat{\xi}_1, \xi_2) \geq 0, j = 1, ..., m\} \geq (1-\bar{\eta})(1-\eta') = 1-\eta$$

where the last inequality is due to $\bar{\eta} = \frac{\eta-\eta'}{1-\eta'}$. Thus, we have the conclusion that $\mathbb{P}_{\overrightarrow{\hat{\xi}_1}}\{x^*(\overrightarrow{\hat{\xi}_1})$ is either undefined or feasible for problem $(1a)\} \geq 1-\beta$. □

5 Numerical experiments

In this section, a simple example and a *supply/demand equilibrium problem* are considered to evaluate numerically the performance of our proposed approaches, i.e., PSAA and PSA methods. All the considered problems are solved using CPLEX 12.6 [CPLEX, 2010] with its default parameters on two different configurations: an Intel(R)D @ 2.00 GHz with 4.0 GB RAM for the simple example and an Intel(R) Core(TM) i7-4600U @ 2.10 GHz 2.70 GHz with 16.0 GB RAM for the *supply/demand equilibrium problem* .

5.1 Comparisons of the proposed methods via a simple example

Let us consider the following simple example:

$$\begin{aligned} \min \quad & c^T x && (25a)\\ s.t. \quad & \mathbb{P}\{\xi_1^T x \geq \xi_2\} \geq 1-\eta && (25b)\\ & x \geq 0, && (25c) \end{aligned}$$

where $c \in R^n$ is a deterministic cost vector, $\xi_1 \in R^n$ and $\xi_2 \in R$ are random and both are assumed uniformly distributed.

Even with only one constraint, problem (25) is still NP-hard as shown by Nemirovski and Shapiro [Nemirovski and Shapiro, 2006] when ξ_1 is independently uniformly distributed in a box and ξ_2 is deterministic.

The assumptions and the parameters are set as follows: $n = 10$, $\eta = 0.1$, c is uniformly generated on the interval $[10, 100]$. ξ_1 and ξ_2 are indepently uniformly distributed. the upper bound of ξ_1 is uniformly generated on the inverval $[10, 20]$ while the lower bound of ξ_1 is uniformly generated on the inverval $[5, 10]$. The upper bound of ξ_2 is uniformly generated on the inverval $[50, 100]$ while the lower bound of ξ_1 is uniformly generated on the inverval $[0, 50]$.

We compared our two proposed methods with SAA approach and the scenario approach. For the bilinear constraint problem with an uniform distribution of ξ_2, the corresponding SA approximation and PSA approximations are linear problems while its SAA approximation is a mixed inter linear problem.

However, PSAA approach problem is not linear as it contains joint probabilistic constraints; but for problem (25) we have a conservative linear approximation as shown by Theorem 5.

The results are reported in Table 2 where column 1 gives the number of scenarios. Columns 2, 3, 4 and 5 present the objective value, the CPU time and the probability threshold of SA, PSA, SAA and PSAA respectively.

No. of Scenario	SA	PSA	SAA	PSAA
N=100	147.84	141.60	118.78	119.05
CPU(seconds)	0.01	0.01	13.67	0.02
Probability	0.9965	0.9963	0.8883	0.8949
N=1000	155.47	151.22	119.42	119.97
CPU(seconds)	0.02	0.02	70.05	0.09
Probability	0.9996	0.9971	0.8917	0.9012
N=10000	157.73	152.65	120.11	120.15
CPU(seconds)	0.05	0.05	10815	0.67
Probability	0.9998	0.9975	0.9031	0.9019

Table 2 Computational results of the simple example.

We can see in Table 2 that SA and PSA have comparable performances in terms of CPU time and the quality of the solutions. This is due to the fact that in both approaches we solve linear programs. Furthermore, PSA solution is better than SA solution, since PSA approach cost is less than SA one. However, the results of SAA and PSAA are different especially in terms of CPU time. In deed, PSAA outperforms SAA as we solve integer linear programs in SAA and continuous ones in PSAA. Additionally, PSAA is slightly more conservative than SAA. To the best of our knowledge, such performances have not been reached so far in sample approximation theory.

5.2 Problem of supply/demand equilibrium under uncertainty

In this section we consider a supply-demand equilibrium problem which is also discussed by Gorge [Gorge, 2013]. This problem is taken from the electrical industry and is a sub-problem of the Unit Commitment Problem (UCP), aiming at minimizing the global production cost while satisfying the supply-demand balance and the operational constraints of a mix of power generation units (hydraulic valleys, nuclear plants and classical thermal units - coal, fuel and gas-) for a discrete time-period.

We propose the following concise formulation that emphasizes the structure of the problem :

$$\begin{array}{ll} \min & \sum_{i=1}^{n}\sum_{t=1}^{T} c_{i,t}x_{i,t} \\ \text{s.t.} & \mathbb{P}\{\sum_{i=1}^{n}\tilde{A}_{i,t}x_{i,t} \geq \tilde{D}_t = \sum_{j=1}^{m} b_{t,j}\tilde{D}_j,\ t = 1,...,T\} \geq 1-\eta \\ & \sum_{t=1}^{T} x_{i,t} \leq r_i T,\ i = 1,...,n \\ & 0 \leq x_{i,t} \leq 1,\ i = 1,...,n,\ t = 1,...,T \end{array} \tag{26}$$

where

- $c_{i,t}$ is the production cost for the plant i at time step t;
- r_i is the maximum proportion that the prescribed plant i will be used over the time-horizon. These constraints represents the necessity of shutting down the plants to proceed to maintenance operations for instance;
- $x_{i,t}$ is the command variable of the plant unit i at time steps t ;
- $x_t = (x_{1,t}, ..., x_{n,t}),\ t = 1, ..., T$;
- $\tilde{A}_t = (\tilde{A}_{1,t}, ..., \tilde{A}_{n,t})$: a random vector representing the availability of the production units at time step t;
- $\tilde{D}_t$: a random variable representing the total demand at time step t;
- $(\tilde{D}_1, ..., \tilde{D}_m)$: a random vector on which $\tilde{D}_t$ depend linearly whereas $b_{t,j}$ is the subsequent coefficient.

For the sake of simplicity, we assume that $\tilde{A}_t$ and $\tilde{D}_j$ are independently uniformly distributed. The lower bound and upper bound of $\tilde{A}_{i,t}$ are uniformly generated on the interval $[20, 60]$ and $[60, 110]$ respectively, whilst the lower bound and upper bound of $\tilde{D}_j$ are uniformly generated on the interval $[10, 30]$ and $[30, 50]$ respectively. The other parameters are set as follows: $n = 30$, $T = 10$, $m = 10$, c is uniformly generated on the interval $[0, 100]$, while r is also uniformly generated on the interval $[0.6, 1]$. $b_{t,j}$ is uniformly generated on the interval $[1, 3]$. Moreover, six confidence parameters are considered, precisely, $\eta = 0.30, 0.25, 0.20, 0.15, 0.10, 0.05$.

Numerical results are given by Table 3 where the columns give the same information results as Table 2 but for six different values of η. We can observe that SA approach has comparable CPU time and slightly better solution than PSA approach when η is larger than 0.85. At the opposite, PSA approach has better solution when $\eta < 0.85$. However, it is totally different case for PSAA and SAA approaches. PSAA results are of different order of magnitude than SAA. As η increases, SAA fails to solve the sample problem especially when N increases. This is mainly true when $\eta = 0.3$ whatever the size of the samples whilst PSAA solves the largest instance, i.e., $N = 10000$ within 10 seconds. Moreover, only instances with $N = 500$ are solved by SAA. When $\eta = 0.25$ and $N = 500$, the CPU time required to solve this instance is

greater than two hours. This is due to the high number of variables y_t which are equal to one. Therefore, there are more feasible solutions to explore when η increases.

$p = 0.95$	SA	PSA	SAA	PSAA	$p = 0.90$	SA	PSA	SAA	PSAA
N=500	1958.7	2087.3	1691.0	1823.8	N=500	1958.7	2050.3	1586.4	1731.2
CPU(S)	0.01	0.01	10.47	0.02	CPU(S)	0.01	0.01	67.23	0.02
Prob	0.951	0.987	0.915	0.936	Prob	0.951	0.983	0.855	0.903
N=1000	2155.4	2254.6	1704.6	1850.9	N=1000	2155.4	2219.5	1594.1	1751.2
CPU(S)	0.05	0.04	691.56	0.79	CPU(S)	0.05	0.04	5013.6	0.47
Pro	0.984	0.992	0.918	0.945	Prob	0.984	0.991	0.860	0.909
N=10000	2404.5	2491.9	–	1874.6	N=1000	2404.5	2474.0	–	1766.7
CPU(S)	0.59	0.60	–	13.97	CPU(S)	0.59	0.51	–	10.31
Prob	0.998	0.999	–	0.957	Prob	0.998	0.999	–	0.921
$p = 0.85$	SA	PSA	SAA	PSAA	$p = 0.80$	SA	PSA	SAA	PSAA
N=500	1958.7	2013.3	1520.2	1665.8	N=500	1958.7	1977.1	1468.9	1619.5
CPU(S)	0.01	0.01	362.25	0.02	CPU(S)	0.01	0.01	965.53	0.04
Prob	0.951	0.979	0.791	0.874	Prob	0.951	0.974	0.757	0.846
N=1000	2155.4	2180.4	–	1681.4	N=1000	2155.4	2142.4	–	1630.7
CPU(S)	0.05	0.03	–	0.53	CPU(S)	0.05	0.03	–	0.46
Prob	0.984	0.987	–	0.882	Prob	0.984	0.983	–	0.847
N=10000	2404.5	2465.6	–	1697.9	N=1000	2404.5	2401.2	–	1645.6
CPU(S)	0.59	0.50	–	10.35	CPU(S)	0.59	0.59	–	9.96
Prob	0.998	0.998	–	0.889	Prob	0.998	0.998	–	0.854
$p = 0.75$	SA	PSA	SAA	PSAA	$p = 0.70$	SA	PSA	SAA	PSAA
N=500	1958.7	1942.1	1424.4	1579.5	N=500	1958.7	1908.2	–	1545.3
CPU(S)	0.01	0.01	8079.3	0.82	CPU(S)	0.01	0.01	–	0.04
Pro	0.951	0.969	0.711	0.820	Prob	0.951	0.962	–	0.791
N=1000	2155.4	2106.9	–	1589.3	N=1000	2155.4	2072.3	–	1553.7
CPU(S)	0.05	0.03	–	0.43	CPU(S)	0.05	0.03	–	0.41
Prob	0.984	0.977	–	0.818	Prob	0.984	0.971	–	0.778
N=10000	2404.5	2369.3	–	1602.7	N=1000	2404.5	2323.1	–	1565.9
CPU(S)	0.59	0.51	–	9.76	CPU(S)	0.59	0.47	–	9.77
Prob	0.998	0.998	–	0.829	Prob	0.998	0.998	–	0.798

Table 3 Computational results of supply/demand problem. "–" indicates that no optimal solutions within two hours

6 Conclusions

In this paper, we present a new sample approximation method for chance constrained problems called Partial SAA (PSAA). We show that our approach enjoys partially the same properties as the standard sample approach. Moreover, we show that PSAA problem is convex in some cases while the standard sample average approximation

(SAA) is a mixed integer problem. Meanwhile, a new definition of concave points is introduced for the first time to the best of our knowledge, and plays an important role in the convexity of the probability distribution. Some results on the concave points are presented as well. Our numerical results show the high performances of PSAA for solving large size instances with up to 10000 samples compared to standard SA and SAA approaches. It is easy to see that PSAA can be used for solving a wide range of stochastic problems using a sample approximation with a highly competitive CPU time and slightly more conservative bounds. In addition, future work related to PSAA approach could lies in different directions, e.g., consider more general models with more general distributions, and relax some considered assumptions.. . .

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